The Indirect Proof of the Beal's Conjecture

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Abstract–The proof of the Fermat's Last Theorem (FLT). The proof of the Goldbach's Conjecture. The indirect proof of the Beal's Conjecture.

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MSC–Primary: 11D41, 11P32; Secondary: 11D61.

I. INTRODUCTION

The cover of this issue of the Bulletin is the frontispiece to a volume of Samuel de Fermat's 1670 edition of Bachet's Latin translation of Diophantus's Arithmetica. This edition includes the marginalia of the editor's father, Pierre de Fermat. Among these notes one finds the elder Fermat's extraordinary comment in connection with the Pythagorean equation $x^2 + y^2 = z^2$ the marginal comment that hints at the existence of a proof (a demonstratio sane mirabilis) of what has come to be known as Fermat's Last Theorem. Diophantus's work had fired the imagination of the Italian Renaissance mathematician Rafael Bombelli, as it inspired Fermat a century later. [8]

The Goldbach's Conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes. [2]

The Beal's Conjecture says that there are no solutions to the equation $A^x + B^y = C^z$ in positive integers A, B, C, x, y, z with $A, B,$ and C being pairwise coprime and all of x, y, z being greater than 2. [1]

If A, B, C **are coprime** (mutually relatively prime), with $A^x + B^y = C^z$, then only one number out of $[A, B, C]$ is even. Only three numbers of the solutions $[A, B, C]$ or of the triples (A, B, C) are solution.

II. THE PROOF OF THE FERMAT'S LAST THEOREM

Theorem 1 (FLT). *For all* $n \in \{3,4,5, \ldots\}$ *and for all* $A, B, C \in \{1,2,3, \ldots\}$: $A^n + B^n \neq C^n$.

Proof. Suppose that for some $n \in \{3,4,5, \ldots\}$ and for some coprime $A, B, C \in \{1,2,3, \ldots\}$: $A^n + B^n = C^n$.

Thus it must be $(A + B > C \wedge A^{n-1} + B^{n-1} > C^{n-1})$.

For some coprime $A, B, C, C - A, C - B, v \in \{1,2,3, \ldots\}$:

$$A - (C - B) = B - (C - A) = 2v > 0 \Longrightarrow (C - B + 2v = A \wedge C - A + 2v = B \wedge A + B - 2v = C). \quad (1)$$

Only one number out of the solutions $[A, B, C]$ is even. Hence we can assume that $A, C - B \in \{1,3,5, \ldots\}$.

Let $\{3,5,7, \ldots\} - \{(2a + b)b: a \in \mathrm{N} \wedge b \in [3,5,7, \ldots]\} = \{3,5,7,11,13,17,19,23,29, \ldots\} = \mathbb{P}$.

Every even number which is not the power of number 2 has odd prime divisor, hence sufficient that we prove FLT for $n = 4$ and for odd prime numbers $n \in \mathbb{P}$. [9]

Lemma 1. *For all coprime* $p, q \in \{1,3,5, \ldots\}$ *and for some coprime* $u, v \in \{1,2,3, \ldots\}$ *such that* $p > q$ *and the number* $u - v$ *is positive and odd*:

$$pq = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 = u^2 - v^2 \Longrightarrow \left[\left(pq, \frac{p^2 - q^2}{2}, \frac{p^2 + q^2}{2}\right) = (u^2 - v^2, 2uv, u^2 + v^2) \wedge p + q = 2u \wedge p - q = 2v \wedge p = u + v \wedge q = u - v \wedge (p + q = 2u = 8,10,12, \ldots) \wedge (p - q = 2v = 2,4,6, \ldots)\right]. \ [5]$$

A. Proof For $n = 4$. Suppose that the equation

$$A^4 + B^4 = C^4 \qquad (2)$$

has primitive solutions $[A, B, C] = [A, B, \sqrt{c}]$.

The number $c \in \{9,25,49, \ldots\}$ and $C \in \{1,3,5, \ldots\}$, inasmuch as $C = \sqrt{c}$. Therefore – For some coprime $a, b \in \{1,2,3, \ldots\}$ such that $a - b$ is positive and odd:

$$[(a^2 + b^2)^2 - (2ab)^2 = (a^2 - b^2)^2 = A^2 \wedge 2(a^2 + b^2)2ab = B^2 \wedge (a^2 + b^2)^2 + (2ab)^2 = C^2 = c \wedge (A^2)^2 + (B^2)^2 = (C^2)^2].$$

On the strength of the lemma 1 we get

$$\left[C = \frac{(2ab)^2 + (2b^2)^2}{2 \cdot 2b^2} = a^2 + b^2 \wedge a^2 + b^2 = \frac{(2ab)^2 - (2b^2)^2}{2 \cdot 2b^2} = a^2 - b^2\right] \in \mathbf{0}. \square$$

B. Proof of Another Hypothesis. Suppose that

$$A^4 + B^4 = c^2 \quad (3)$$

has primitive solutions $[A, B, c]$.

We assume that the number c is minimal. [9]

The hypothesis (2) and (3) are different [3], inasmuch as the number $c \in \{3,5,7,\dots\} \setminus \{9,25,49,\dots\}$, with $C \in \left\{\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \sqrt{15}, \sqrt{17}, \sqrt{19}, \sqrt{21}, \dots\right\}$.

On the strength of the lemma 1 we obtain –

For some coprime $p, q \in \{1,3,5,\dots\}$ and for some coprime $U, V \in \{1,2,3,\dots\}$ and for some coprime $a, b \in \{1,2,3,\dots\}$ and for some pairwise relatively prime $x, y, z \in \{1,2,3,\dots\}$ such that $p > q$ and the numbers $U - V, x - y, a - b$ are positive and odd:

$$\Big[(pq)^2 = \frac{(p^2+q^2)^2}{4} - \frac{(p^2-q^2)^2}{4} = U^2 - V^2$$
$$= (a^2 - b^2)^2 = (a^2 + b^2)^2 - (2ab)^2$$
$$= A^2 \wedge p = a + b \wedge q$$
$$= a - b \wedge \frac{p^4 - q^4}{2}$$
$$= \frac{p^2 + q^2}{2}(p^2 - q^2) = 2UV$$
$$= 2(a^2 + b^2)2ab = B^2 \wedge \frac{p^4 + q^4}{2}$$
$$= \frac{(p^2+q^2)^2}{4} + \frac{(p^2-q^2)^2}{4} = U^2 + V^2$$
$$= (a^2 + b^2)^2 + (2ab)^2 = c$$
$$= C^2 \wedge \frac{p^2 + q^2}{2} = U = a^2 + b^2$$
$$= z^2 \wedge \frac{p^2 - q^2}{2} = V = 2ab \wedge 4ab$$
$$= (2xy)^2 \wedge a = x^2 \wedge b$$
$$= y^2 \wedge x^4 + y^4 = z^2 < c^2\Big] \Longrightarrow z < c,$$

which is inconsistent with minimal c. □

C. Proof For $n \in \mathbb{P}$. Without loss for this proof we can assume that $4 \nmid B, C$. In view of (1) we will have –

For some $n \in \mathbb{P}$ and for some $C, B, C - A \in \{1,2,3,\dots\}$ and for some $C - B, A, \nu \in \{1,3,5,\dots\}$:

$$\Big[(C - B + 2\nu)^n = (C - B + B)^n - B^n$$
$$\Longrightarrow (C - B)^{n-2}\nu + (n-1)(C-B)^{n-3}\nu^2 + \cdots + 2^{n-2}\nu^{n-1} + \frac{2^{n-1}\nu^n}{n(C-B)}$$
$$= \frac{B}{2}\Big[(C-B)^{n-2} + \frac{n-1}{2}(C-B)^{n-3}B + \cdots + B^{n-2}\Big] \wedge n \mid \nu$$
$$\wedge \ (n \mid B \veebar n \mid C)\Big] \wedge$$

$$\Big[(C - A + 2\nu)^n = (C - A + A)^n - A^n \Longrightarrow (C - A)^{n-2}2\nu + \frac{n-1}{2}(C-A)^{n-3}(2\nu)^2 + \cdots + (2\nu)^{n-1} + \frac{(2\nu)^n}{n(C-A)}$$
$$= A\Big[(C-A)^{n-2} + \frac{n-1}{2}(C-A)^{n-3}A + \cdots + A^{n-2}\Big] \wedge n \mid \nu$$
$$\wedge \ (n \mid A \veebar n \mid C)\Big] \wedge$$

$$\Big[A^n + B^n = C^n = (A + B - 2\nu)^n$$
$$= A^n + nA^{n-1}B + \frac{n(n-1)}{2}A^{n-2}B^2 + \cdots + nAB^{n-1} + B^n$$
$$+ n(A+B)^{n-1}(-2\nu) + \frac{n(n-1)}{2}(A+B)^{n-2}(-2\nu)^2 + \cdots + n(A+B)(-2\nu)^{n-1} + (-2\nu)^n$$
$$\Longrightarrow \Big[0 = AB\frac{\left(A^{n-2} + \frac{n-1}{2}A^{n-3}B + \cdots + B^{n-2}\right)}{A+B}$$
$$+ (A+B)^{n-2}(-2\nu) + \frac{n-1}{2}(A+B)^{n-3}(-2\nu)^2 + \cdots + (-2\nu)^{n-1} + \frac{(-2\nu)^n}{n(A+B)} \wedge n \mid \nu$$
$$\wedge \left(n \mid A \veebar n \mid B \veebar (n^{n-1} \mid A + B \wedge n \mid C)\right)\Big]\Big].$$

If $n \mid A \equiv 1$, then

$$\left[(n \mid A \veebar n \mid C) \equiv 1 \wedge (n \mid B \veebar n \mid C) \equiv 0 \wedge \left(n \mid A \veebar n \mid B \veebar (n^{n-1} \mid A + B \wedge n \mid C)\right) \equiv 1\right] \in \mathbf{0}.$$

If $n \mid B \equiv 1$, then

$$\big[(n \mid A \veebar n \mid C) \equiv 0 \wedge (n \mid B \veebar n \mid C) \equiv 1 \wedge \big(n \mid A \veebar n \mid B \veebar (n^{n-1} \mid A + B \wedge n \mid C)\big) \equiv 1\big] \in \mathbf{0}.$$

If $n \mid C \equiv 1$, then

$$\big[(n \mid A \veebar n \mid C) \equiv 1 \wedge (n \mid B \veebar n \mid C) \equiv 1 \wedge \big(n \mid A \veebar n \mid B \veebar (n^{n-1} \mid A + B \wedge n \mid C)\big) \equiv 1\big] \in \mathbf{1}.$$

For some $n \in \mathbb{P}$ and for some $e, m, c, h \in \{1,3,5,\dots\}$ such that n, e, m, c and h are coprime:

$$[nemch = v \wedge n \nmid emch \wedge (h^n = C - A \vee (2h)^n = C - A) \wedge c^n = C - B].$$

B. 1. Proof For Odd $A, B, C - B$.

For some $n \in \mathbb{P}$ and for some $e, m, c, h \in \{1,3,5,\dots\}$ such that n, e, m, c and h are coprime:

$$[c^n + 2nemch = A \wedge h^n + 2nemch = B \wedge 2^n n^{n-1} m^n = A + B = c^n + h^n + 4nemch \wedge c^n + B = C] \Rightarrow [2^n n^{n-1} m^n = c^n + h^n + 4nemch \wedge n \mid (c + h - h)^n + h^n] \Rightarrow (n \mid c + h \wedge n^2 \mid c^n + h^n \wedge n \mid emch),$$

which is inconsistent with $n \nmid emch$. □

B. 2. Proof For Even $B, C - A$.

For some $n \in \mathbb{P}$ and for some $e, m, c, h \in \{1,3,5,\dots\}$ such that n, e, m, c and h are coprime:

$$[c^n + 2nemch = A \wedge (2h)^n + 2nemch = B \wedge n^{n-1} m^n = A + B = c^n + (2h)^n + 4nemch \wedge c^n + B = C] \Rightarrow [n^{n-1} m^n = c^n + (2h)^n + 4nemch \wedge n \mid (c + 2h - 2h)^n + (2h)^n] \Rightarrow (n \mid c + 2h \wedge n^2 \mid c^n + (2h)^n \wedge n \mid emch),$$

which is inconsistent with $n \nmid emch$.

This is the proof.

III. The Proof of The Golbdach's Conjecture

Conjecture 1. For each $u \in \{2,3,4,\dots\}$ there exist $p, q \in \mathbb{P} \cup \{2\}$: $2u = p + q$.

Proof. $4 = 2 + 2, 6 = 3 + 3$. For each $u \in \{4,5,6,\dots\}$ there exists $v \in \{1,2,3,\dots\}$ and there exist $p, q \in \mathbb{P}$:

$$[\mathbf{gcd}(u, v) = 1 \wedge u + v = p \wedge u - v = q \wedge 2u = p + q \wedge 2v = p - q].$$

This is the proof.

IV. The Indirect Proof of The Beal's Conjecture

Conjecture 1. For all $x, y, z \in \{3,4,5,\dots\}$ the equation

$$A^x + B^y = C^z \qquad (4)$$

has no primitive solutions.

Proof. Suppose that the equation (4) has primitive solutions, with $\neg x = y = z$ – because FLT is true. [4], [6] and [7]

Then with (4) we will have

$$[(A^x + B^x < C^x \wedge A^y + B^y < C^y \wedge A^z + B^z < C^z) \veebar (A^x + B^x > C^x \wedge A^y + B^y > C^y \wedge A^z + B^z > C^z)]. \qquad (5)$$

Let consequtively **in each finally primitive triple** (A, B, C) there exists the common prime factor 2^k and the numbers A, B and C are inviolable.

$$(5) \Rightarrow$$

$$\big[\big(A^{x+a} + B^x = C^x \Rightarrow (2^x A, 2^{x+a} B, 2^{x+a} C) \Rightarrow (2^{-a} A, B, C)\big) \veebar \big(A^x + B^x = C^{x+a} \Rightarrow (2^{x+a} A, 2^{x+a} B, 2^x C) \Rightarrow (A, B, 2^{-a} C)\big)\big],$$

which is inconsistent with A, B and C are inviolable.

However also from

$$(5) \Rightarrow$$

$$[(A^x + B^x = C^{x-a} \wedge A^y + B^y = C^{y-b} \wedge A^z + B^z = C^{z-c}) \veebar (A^{x-a} + B^x = C^x \wedge A^{y-b} + B^y = C^y \wedge A^{z-c} + B^z = C^z)] \Rightarrow [(a = b = c \wedge x = y = z) \veebar (a = b = c \wedge x = y = z)],$$

which is inconsistent with $\neg x = y = z$.

This is the proof.

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