

The Generalization of Fermat's Last Theorem

By Leszek W. Guła

Lublin-POLAND

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Abstract—

The proof of Fermat's Last Theorem (FLT). The proof of Jeśmanowicz's Conjecture. The proof of Goldbach's Conjecture. The Beal's Conjecture 1&2.

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I. INTRODUCTION

Among these notes one finds the elder Fermat's extraordinary comment in connection with the Pythagorean equation the marginal comment that hints at the existence of a proof (a demonstratio sane mirabilis) of what has come to be known as Fermat's Last Theorem. [4]

Jeśmanowicz's Conjecture [1] concerns the Diophantus Equation.

The Goldbach's Conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes. [7]

Beal's Conjecture is a generalization of Fermat's Last Theorem. [6]

II. THE PROOF OF FERMAT'S LAST THEOREM

Theorem 1 (LWG Theorem 1999).

$$\{(2a + b)b: a \in [0, 1, 2, \dots] \wedge b \in [3, 5, 7, \dots]\} = \{9, 15, 21, 25, 27, 33, 35, 39, 45, 49, \dots\} \Rightarrow \\ \{3, 5, 7, \dots\} - \{9, 15, 21, 25, 27, 33, 35, 39, 45, 49, \dots\} = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \dots\} = \mathbb{P}. [2]$$

Theorem 2 (FLT). For all $n \in \{3, 4, 5, \dots\}$ and for all $A, B, C \in \{1, 2, 3, \dots\}$: $A^n + B^n \neq C^n$.

Proof. Suppose that for some $n \in \{3, 4, 5, \dots\}$ and for some co-prime $A, B, C \in \{1, 2, 3, \dots\}$: $A^n + B^n = C^n$.

Remark 1. Fermat did not proved his own theorem for $n = 4$. [5]

These hypothesis $A^4 + B^4 = C^4$ and $A^4 + B^4 = c^2$ are different because

$$[(A^4 + B^4 = C^4 \Rightarrow C = \sqrt{C}) \wedge (A^4 + B^4 = c^2 \Rightarrow c = C^2)],$$

inasmuch as a hypothetical solutions are different, namely $[A, B, C] \neq [A, B, c]$.

Sufficient that we prove FLT for $n = 4$ and for $n \in \mathbb{P}$. [5] This is the remark 1.

For some $n \in \{3, 4, 5, \dots\}$ and for some $A, B, C \in \{1, 2, 3, \dots\}$:

$$(A + B \leq C \wedge A^2 + B^2 \leq C^2 \wedge \dots \wedge A^{n-1} + B^{n-1} \leq C^{n-1}) \Rightarrow A^n + B^n < C^n.$$

Hence it must be $(A + B > C \wedge A^2 + B^2 > C^2 \wedge \dots \wedge A^{n-1} + B^{n-1} > C^{n-1})$. [3]

Without loss for this proof we can assume that $\{A, (B - C) \in [1, 3, 5, \dots] \wedge (4 \nmid B \vee 4 \nmid C)\}$.

A. Proof For $n = 4$.

For some $A, C, v \in \{1, 3, 5, \dots\}$ and for some $B \in \{6, 10, 14, \dots\}$ such that A, C and B are co-prime:

$$\left\{ \begin{aligned} [A - (C - B) = 2v \wedge A^2 + B^2 > C^2 \wedge C - B + 2v = A \wedge C - A + 2v = B \wedge (C - B) + (C - A) + 2v \\ = C \wedge (C - A + 2v)^4 = (C - A + A)^4 - A^4 \wedge (C - B + 2v)^4 = (C - B + B)^4 - B^4] \\ \Rightarrow \left[(C - A)^2 2v + \frac{3}{2}(C - A)(2v)^2 + (2v)^3 + \frac{4v^4}{C - A} \right. \\ = (C - A)^2 A + \frac{3}{2}(C - A)A^2 + A^3 \wedge (C - B)^2 2v + \frac{3}{2}(C - B)(2v)^2 + (2v)^3 + \frac{4v^4}{C - B} \\ \left. = (C - B)^2 B + \frac{3}{2}(C - B)B^2 + B^3 \wedge 2v^2 > (C - A)(C - B) \right] \end{aligned} \right\}.$$

Thus

$$\left\{ \frac{4v^4}{C-A}, \frac{4v^4}{C-B} \in [1,3,5, \dots] \wedge 2v^2 > (C-A)(C-B) \right\}.$$

Hence – For some $c, d, v \in \{1,3,5, \dots\}$ such that c, d are co-prime:

$$(c^4 = C - B \wedge 4d^4 = C - A \wedge c^4 + 2v = A \wedge 4d^4 + 2v = B \wedge v^2 > 2c^4d^4) \Rightarrow v > \sqrt{2}(cd)^2.$$

Therefore – For some $c, d, e \in \{1,3,5, \dots\}$ such that c, d and e are co-prime: $cde = v$.

Further it must be – For some $c, d, e, A \in \{1,3,5, \dots\}$ such that c, d and e are co-prime:

$$\begin{aligned} (2cde + 4d^4)^4 &= [(4d^4 + A)^2]^2 - (A^2)^2 = [(4d^4 + A)^2 + A^2](2d^4 + A)8d^4 \Rightarrow \\ 2(ce + 2d^3)^4 &= [(4d^4 + A)^2]^2 - (A^2)^2 = [(4d^4 + A)^2 + A^2](2d^4 + A). \end{aligned}$$

We assume that for some co-prime $z, w, x \in \{1,3,5, \dots\}$ and for some $y \in \{6,10,14, \dots\}$:

$$\begin{aligned} \{zw = ce + 2d^3 \wedge x + y = 2d^4 + A + 2d^4 \wedge x = 2d^4 + A \wedge y = 2d^4 \wedge 2(zw)^4 = [(x+y)^2 + (x-y)^2]x \\ = 2(x^2 + y^2)x \wedge z^4w^4 = (x^2 + y^2)x \wedge (z^2)^2 = x^2 + y^2 \wedge w^4 = x\} \Rightarrow 4 \mid y, \end{aligned}$$

which is inconsistent with $4 \nmid y$. [2]

This is the proof for $n = 4$.

B. Proof For $n \in \mathbb{P}$. In [2] we have the proof of FLT for $n \in \mathbb{P}$.

Theorem 3 (LWG Theorem 03&04 June1997). For each $g \in \{8,12,16, \dots\}$ or for each $g \in \{3,5,7, \dots\}$ there exist finitely many pairs (s, t) of positive integers such that:

$$g = \left(\frac{g+d^2}{2d} \right)^2 - \left(\frac{g-d^2}{2d} \right)^2 = s^2 - t^2 = (s+t)(s-t) = \frac{g}{d}(s-t) = \frac{g}{d}d = g,$$

where $d \mid g$ and $d < \sqrt{g}$ and $d, \frac{g}{d} \in \{2,4,6, \dots\}$ with even g or $d \in \{1,3,5, \dots\}$ with odd g . [2]

Theorem 4. For all $x, u, v \in \{1,2,3, \dots\}$ such that $\mathbf{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$:

$$\begin{aligned} \left\{ (u+v)^x(u-v)^x = \left[\frac{(u+v)^x + (u-v)^x}{2} \right]^2 - \left[\frac{(u+v)^x - (u-v)^x}{2} \right]^2 \wedge (u^2 - v^2)^{2+x} + (2uv)^{2+x} \right. \\ = (u^2 - v^2)^2(u^2 - v^2)^x + (2uv)^2(2uv)^x < (u^2 - v^2)^2(u^2 + v^2)^x + (2uv)^2(u^2 + v^2)^x \\ \left. = (u^2 + v^2)^{2+x} \right\} \Rightarrow (u^2 - v^2, 2uv, u^2 + v^2). [2] \end{aligned}$$

Theorem 5. Let u and v be two relatively prime natural numbers such that $u - v$ is positive and odd. Then – For any primitive Pythagorean triple (x, y, z) there exists different and only one shared pair (u, v) , which gives only one primitive Diophantus triple $(u^2 - v^2, 2uv, u^2 + v^2) = (x, y, z)$.

III. THE PROOF OF JEŚMANOWICZ'S CONJECTURE

Conjecture 1 (Jeśmanowicz Conjecture). For all $x, y, z, u, v \in \{1,2,3, \dots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\mathbf{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$: $(u^2 - v^2)^x + (2uv)^y \neq (u^2 + v^2)^z$.

Proof. Suppose that for some $x, y, z, u, v \in \{1,2,3, \dots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\mathbf{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$: $(u^2 - v^2)^x + (2uv)^y = (u^2 + v^2)^z$.

On the strength of the Theorems 3,4 and 5 we will have – For some $x, y, z, u, v \in \{1,2,3, \dots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\mathbf{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$:

$$\begin{aligned} [(u^2 - v^2)^x + (2uv)^y = (u^2 + v^2)^z \wedge u^2 - v^2 + 2uv > u^2 + v^2 \wedge (u^2 - v^2)^2 + (2uv)^2 = (u^2 + v^2)^2] \\ \Rightarrow [(u^2 - v^2)^x = (u^2 - v^2)^z \wedge (2uv)^y = (2uv)^2 \wedge (u^2 + v^2)^z = (u^2 + v^2)^2] \Rightarrow \\ (x, y, z) = (2,2,2), \end{aligned}$$

which is inconsistent with $(x, y, z) \neq (2,2,2)$. [2]

This is the proof.

IV. THE PROOF OF GOLDBACH'S CONJECTURE

Conjecture 2 (Goldbach Conjecture). For all $Z \in \{6,8,10, \dots\}$ and for some $X, Y \in \mathbb{P}: Z = X + Y$.

Proof. The key of this proof are two common prime factors: 2 and 3.

$$\{18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, \dots\} =$$

$$\{(3Z): (3Z) = (3X) + (3Y) \wedge 3X \leq 3Y \wedge X, Y \in \mathbb{P} \wedge (3X), (3Y) \in \{9, 15, 21, 27, 33, 39, 45, 51, \dots\}\} \Rightarrow$$

$$\{6, 8, 10, 12, 14, 16, 18, 20, 22, \dots\} = \{Z: Z = X + Y \wedge X \leq Y \wedge X, Y \in \mathbb{P}\}. [2]$$

It is easy to verify that

$$\begin{aligned} [18 = 9 + 9, 24 = 9 + 15, 30 = 9 + 21 = 15 + 15, 36 = 15 + 21, 42 = 9 + 33 = 21 + 21, 48 = 9 + 39 \\ = 15 + 33, 54 = 15 + 39 = 21 + 33, 60 = 9 + 51 = 21 + 39, 66 = 9 + 57 = 15 + 51 \\ = 33 + 33, 72 = 15 + 57 = 21 + 51 = 33 + 39, 78 = 9 + 69 = 21 + 57 = 39 + 39, \dots]. \end{aligned}$$

Thus we get all equations according to the Goldbach's Conjecture, namely

$$\begin{aligned} [6 = 3 + 3, 8 = 3 + 5, 10 = 3 + 7 = 5 + 5, 12 = 5 + 7, 14 = 3 + 11 = 7 + 7, 16 = 3 + 13 = 5 + 11, 18 \\ = 5 + 13 = 7 + 11, 20 = 3 + 17 = 7 + 13, 22 = 3 + 19 = 5 + 17 = 11 + 11, 24 = 5 + 19 \\ = 7 + 17 = 11 + 13, 26 = 3 + 23 = 7 + 19 = 13 + 13, 28 = 5 + 23 = 11 + 17, 30 \\ = 7 + 23 = 11 + 19 = 13 + 17, 32 = 3 + 29 = 13 + 19, \dots]. \end{aligned}$$

This is the proof.

V. THE BEAL'S CONJECTURE 1&2

Conjecture 3 (Beal Conjecture 1). For some $x, y, z \in \{3, 4, 5, \dots\}$ and for some $A, B, C \in \{1, 2, 3, \dots\}$ such that A, B and C have the common prime factor $p \geq 2$:

$$A^x + B^y = C^z.$$

The direct proof in [2] is the complete proof of Beal's Conjecture 1.

Beal Conjecture 2. For all $x, y, z \in \{3, 4, 5, \dots\}$ the equation

$$A^x + B^y = C^z$$

has no primitive solutions $[A, B, C]$ in $\{1, 2, 3, \dots\}$.

Proof. Suppose that for some $x, y, z \in \{3, 4, 5, \dots\}$ the equation

$$A^x + B^y = C^z$$

has primitive solutions $[A, B, C]$ in $\{1, 2, 3, \dots\}$.

Then only one number out of any solution $[A, B, C]$ is even.

From the above it follows that

$$[(A + B > C \wedge A^2 + B^2 > C^2) \vee (A + B = C \wedge A^2 + B^2 < C^2) \vee (A + B < C \wedge A^2 + B^2 < C^2)].$$

Remark 2. Let $\mathit{cpf}(p4, p5, p3) = p$, where p is the odd common prime factor with the numbers of the primitive solution $[4, 5, 3]$ of the equation $4 + 5 = 3^2 \equiv 1$.

If $2 > 1 = 1$, then for $p = \mathit{cpf}(p, 5) = 5: p4 + p5 = p^2 3^2 \equiv 0$.

Hence the indirect proof of Beal Conjecture 2 in [2] is the false proof. This is the remark 2.

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