The Generalization of Fermat's Last Theorem

By Leszek W. Guła

Lublin-POLAND

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Abstract—
About the proper proof of Fermat's Last Theorem.
The proper proof of Jeśmanowicz's Conjecture.
The proper proof of Beal's Conjecture 2.

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I. INTRODUCTION

Fermat's Last Theorem is the famous theorem.
Jeśmanowicz's Conjecture [1] concerns the Diophantus Equation.
Beal's Conjecture is a generalization of Fermat's Last Theorem. [4]

II. ABOUT THE PROPER PROOF OF FERMAT'S LAST THEOREM

Theorem 1 (Femat Last Theorem). *For all* $n \in \{3,4,5,\dots\}$ *and for all* $A, B, C \in \{1,2,3,\dots\}$:

$$A^n + B^n \neq C^n.$$

Proof in [2].

If $A^n + B^n = C^n$, then $A + B - C > 0$ and $A^2 + B^2 > C^2$, where A, B and C are co-prime.

Only one number out of each solution $[A, B, C]$ is even. $A + B - C \in [2,4,6,\dots]$ and $A, C - B \in \{1,3,5,\dots\}$.

We assume – For some co-prime $u, v \in \{1,2,3,\dots\}$ such that $u - v \in \{1,3,5,\dots\}$: $A + B - C = 2v(u - v)$.

$[A = C - B + 2v(u - v) \wedge B = C - A + 2v(u - v) \wedge C = C - B + C - A + 2v(u - v) \wedge A^2 + B^2 > C^2]$.

Thus $2v^2(u - v)^2 > (C - A)(C - B)$, with $[(n \mid v \wedge n \mid A) \vee (n \mid v \wedge n \mid B) \vee (n \mid v \wedge n \mid C)]$.

If $A = u^2 - v^2$, then $n \nmid A$. Then the proof is incomplete.

If $A = (u - v)^n + 2v(u - v)$, then $n \nmid A$. [3] Then the proof is incomplete.

If $C = u^2 + v^2$, then $n \nmid C$. Then the proof is incomplete.

If $B = 2uv$, then $(A = u^2 + a - v^2 \wedge C = u^2 + a + v^2 \wedge n \mid v \wedge n \nmid C, A)$. Then the proof is incomplete.

From the Fermat Equation it follows that – For any primitive solution $[A, B, C]$ and for all $u, v \in \{1,2,3,\dots\}$ such that $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5,\dots\}$:

$$\{\neg[(A = u^2 - v^2 \wedge B = 2uv \wedge C = u^2 + v^2) \equiv 0 \Longrightarrow A + B - C = 2v(u - v) \equiv 1] \\ \equiv 0 \wedge \neg[(A \neq u^2 - v^2 \wedge B \neq 2uv \wedge C \neq u^2 + v^2) \equiv 1 \Longrightarrow A + B - C \neq 2v(u - v) \equiv 0] \\ \equiv 1\} \equiv 0.$$

We have $[(A \neq u^2 - v^2 \wedge B \neq 2uv \wedge C \neq u^2 + v^2) \wedge A + B - C = 2v(u - v)] \equiv 1$.

If $n = 4$, then for some $c, d, e \in \{1,3,5,\dots\}$ such that c, d and e are co-prime: $cde = v(u - v) = vc = v$.

If n is odd prime number, then for some co-prime $e, m, c, h \in \{1,3,5,\dots\}$: $nemch = v(u - v) = vc = v$.

Further in [2] we have the proper proof of Fermat's Last Theorem.

Theorem 2 (LWG Theorem 03&04 June1997). *For each* $g \in \{8,12,16,\dots\}$ *or for each* $g \in \{3,5,7,\dots\}$ *there exist finitely many pairs* (s, t) *of positive integers such that*:

$$g = \left(\frac{g + d^2}{2d}\right)^2 - \left(\frac{g - d^2}{2d}\right)^2 = s^2 - t^2 = (s + t)(s - t) = \frac{g}{d}(s - t) = \frac{g}{d}d = g,$$

where $d|g$ *and* $d < \sqrt{g}$ *and* - $d, \frac{g}{d} \in \{2,4,6,\dots\}$ *with even* g *or* $d \in \{1,3,5,\dots\}$ *with odd* g. [2]

Theorem 3. *For all* $x,u,v \in \{1,2,3,...\}$ *such that* $\boldsymbol{gcd}(u,v)=1$ *and* $u-v \in \{1,3,5,...\}$:

$$\begin{cases}(u+v)^x(u-v)^x = \left[\frac{(u+v)^x+(u-v)^x}{2}\right]^2 - \left[\frac{(u+v)^x-(u-v)^x}{2}\right]^2 \wedge (u^2-v^2)^{2+x}+(2uv)^{2+x} \\ = (u^2-v^2)^2(u^2-v^2)^x+(2uv)^2(2uv)^x < (u^2-v^2)^2(u^2+v^2)^x+(2uv)^2(u^2+v^2)^x \\ = (u^2+v^2)^{2+x}\end{cases}. \text{ [2]}$$

Definition 1. $\boldsymbol{cpf}[\boldsymbol{p}(u^2-v^2),\boldsymbol{p}2uv,\boldsymbol{p}(u^2+v^2)]=\boldsymbol{p}$, where $\boldsymbol{p}$ is the odd common prime factor $(\boldsymbol{cpf})$ with the numbers of the solutions $[u^2-v^2,2uv,u^2+v^2]$ such that $\boldsymbol{p},uv,u^2-v^2$ are co-prime. [2]

This is the definition 1.

Definition 2. $\boldsymbol{cpf}(\boldsymbol{p}A,\boldsymbol{p}B,\boldsymbol{p}C)=\boldsymbol{p}$, where $\boldsymbol{p}$ is the odd common prime factor $(\boldsymbol{cpf})$ with the numbers of the primitive solutions $[A,B,C]$. [2]

This is the definition 2.

III. The Proper Proof Of Jeśmanowicz's Conjecture

Conjecture 1 (Jeśmanowicz Conjecture). For all $x,y,z,u,v \in \{1,2,3,...\}$ such that $(x,y,z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u,v)=1$ and $u-v \in \{1,3,5,...\}$: $(u^2-v^2)^x+(2uv)^y \neq (u^2+v^2)^z$.

Proof. Suppose that for some $x,y,z,u,v \in \{1,2,3,...\}$ such that $(x,y,z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u,v)=1$ and $u-v \in \{1,3,5,...\}$: $(u^2-v^2)^x+(2uv)^y = (u^2+v^2)^z$.

If $u=2$ and $v=1$, then

$$(3+4>5 \wedge 3^1+4^2<5^2 \wedge 3^2+4^1<5^2 \wedge 3^1+4^3>5^2 \wedge 3^3+4^1>5^2 \wedge 3^3+4^2<5^3 \wedge 3^2+4^3<5^3).$$

If $u-v>v$, then $u^2-v^2+2uv>u^2+v^2$ and

$$[(u^2-v^2)^1+(2uv)^2<(u^2+v^2)^2 \wedge (u^2-v^2)^2+(2uv)^1<(u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^1+(2uv)^3>(u^2+v^2)^2 \wedge (u^2-v^2)^3+(2uv)^1>(u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^3+(2uv)^2<(u^2+v^2)^3 \wedge (u^2-v^2)^2+(2uv)^3<(u^2+v^2)^3].$$

If $u-v<v$, then $u^2-v^2+2uv>u^2+v^2$ and

$$[(u^2-v^2)^1+(2uv)^2<(u^2+v^2)^2 \wedge (u^2-v^2)^2+(2uv)^1<(u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^1+(2uv)^3>(u^2+v^2)^2 \wedge (u^2-v^2)^3+(2uv)^1<(u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^3+(2uv)^2<(u^2+v^2)^3 \wedge (u^2-v^2)^2+(2uv)^3<(u^2+v^2)^3].$$

On the strength of the Theorem 3 – For some $z \in \{3,4,5,...\}$ and for some $p,q \in \{0,1,2,...\}$ and for some $u,v \in \{1,2,3,...\}$ such that $p+q>0$ and $u-v \in \{1,3,5,...\}$ and $\boldsymbol{p},uv,u-v$ are co-prime:

$$[(u^2-v^2)^{z+p}+(2uv)^{z-q}=(u^2+v^2)^z \vee (u^2-v^2)^{z-p}+(2uv)^{z+q}=(u^2+v^2)^z].$$

Then for some $\boldsymbol{p}$ it must be $\boldsymbol{p}=\boldsymbol{cpf}(\boldsymbol{p},u^2+v^2)$.

If $z+p>z\geq z-q$, then for some $\boldsymbol{p}=\boldsymbol{cpf}(\boldsymbol{p},u^2+v^2)$:

$$[\boldsymbol{p}^{z+p}(u^2-v^2)^{z+p}+\boldsymbol{p}^{z-q}(2uv)^{z-q}=\boldsymbol{p}^z(u^2+v^2)^z] \Rightarrow$$
$$\{[\boldsymbol{p}^{p+q}(u^2-v^2)^{z+p}+(2uv)^{z-q}=\boldsymbol{p}^q(u^2+v^2)^z \vee \boldsymbol{p}^p(u^2-v^2)^{z+p}+(2uv)^z=(u^2+v^2)^z]\}$$
$$\Rightarrow [\boldsymbol{cpf}(\boldsymbol{p},(2uv)^{z-q})>1 \vee \boldsymbol{cpf}(\boldsymbol{p},(2uv)^z)>1],$$

which is inconsistent with $\boldsymbol{p},uv,u^2-v^2$ are co-prime.

If $z+q>z\geq z-p$, then for some $\boldsymbol{p}=\boldsymbol{cpf}(\boldsymbol{p},u^2+v^2)$:

$$[\boldsymbol{p}^{z-p}(u^2-v^2)^{z-p}+\boldsymbol{p}^{z+q}(2uv)^{z+q}=\boldsymbol{p}^z(u^2+v^2)^z] \Rightarrow$$
$$\{[(u^2-v^2)^{z-p}+\boldsymbol{p}^{p+q}(2uv)^{z+q}=\boldsymbol{p}^p(u^2+v^2)^z \vee (u^2-v^2)^{z-p}+\boldsymbol{p}^q(2uv)^{z+q}=(u^2+v^2)^z]\}$$
$$\Rightarrow [\boldsymbol{cpf}(\boldsymbol{p},(u^2-v^2)^{z-p})>1 \vee \boldsymbol{cpf}(\boldsymbol{p},(u^2-v^2)^{z-p})>1],$$

which is inconsistent with $\boldsymbol{p},uv,u^2-v^2$ are co-prime. This is the proper proof.

IV. THE PROPER PROOF OF BEAL'S CONJECTURE

Conjecture 2 (Beal Conjecture 2). For all $x, y, z \in \{3,4,5 \dots\}$ the equation

$$A^x + B^y = C^z$$

has no primitive solutions $[A, B, C]$ in $\{1,2,3, \dots\}$.

Proof. Let for some $x, y, z \in \{3,4,5, \dots\}$ and for some coprime $A, B, C \in \{1,2,3, \dots\}$: $A^x + B^y = C^z$.

Then only one number out of each solution $[A, B, C]$ is even. Thus we assume that A is odd.

If $y > x = z$, then for some $\boldsymbol{p} = \boldsymbol{cpf}(\boldsymbol{p}, A)$:

$$(\boldsymbol{p}^x A^x + \boldsymbol{p}^y B^y = \boldsymbol{p}^z C^z) \Longrightarrow (A^x + \boldsymbol{p}^{y-z} B^y = C^z) \Longrightarrow \boldsymbol{cpf}(\boldsymbol{p}, C^z) > 1,$$

which is inconsistent with the primitive solution $[A, B, C]$.

If $y > x > z$, then for some $\boldsymbol{p} = \boldsymbol{cpf}(\boldsymbol{p}, A)$:

$$(\boldsymbol{p}^x A^x + \boldsymbol{p}^y B^y = \boldsymbol{p}^z C^z) \Longrightarrow (\boldsymbol{p}^{x-z} A^x + \boldsymbol{p}^{y-z} B^y = C^z) \Longrightarrow \boldsymbol{cpf}(\boldsymbol{p}, C^z) > 1,$$

which is inconsistent with the primitive solution $[A, B, C]$.

If $y > z > x$, then for some $\boldsymbol{p} = \boldsymbol{cpf}(\boldsymbol{p}, C)$:

$$(\boldsymbol{p}^x A^x + \boldsymbol{p}^y B^y = \boldsymbol{p}^z C^z) \Longrightarrow (A^x + \boldsymbol{p}^{y-x} B^y = \boldsymbol{p}^{z-x} C^z) \Longrightarrow \boldsymbol{cpf}(\boldsymbol{p}, A^z) > 1,$$

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If $z > y = x$, then for some $\boldsymbol{p} = \boldsymbol{cpf}(\boldsymbol{p}, A)$:

$$(\boldsymbol{p}^x A^x + \boldsymbol{p}^y B^y = \boldsymbol{p}^z C^z) \Longrightarrow (A^x + B^y = \boldsymbol{p}^{z-x} C^z) \Longrightarrow \boldsymbol{cpf}(\boldsymbol{p}, B^z) > 1,$$

which is inconsistent with the primitive solution $[A, B, C]$.

If $z > x > y$, then for some $\boldsymbol{p} = \boldsymbol{cpf}(\boldsymbol{p}, A)$:

$$(\boldsymbol{p}^x A^x + \boldsymbol{p}^y B^y = \boldsymbol{p}^z C^z) \Longrightarrow (\boldsymbol{p}^{x-y} A^x + B^y = \boldsymbol{p}^{z-y} C^z) \Longrightarrow \boldsymbol{cpf}(\boldsymbol{p}, B^z) > 1,$$

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$$(\boldsymbol{p}^x A^x + \boldsymbol{p}^y B^y = \boldsymbol{p}^z C^z) \Longrightarrow (\boldsymbol{p}^{x-y} A^x + B^y = \boldsymbol{p}^{z-y} C^z) \Longrightarrow \boldsymbol{cpf}(\boldsymbol{p}, B^z) > 1,$$

which is inconsistent with the primitive solution $[A, B, C]$. This is the proper proof.

REFERENCES

[1] Bobiński, Z., Kamiński, B.: WIADOMOŚCI MATEMATYCZNE XXXV, SERIES II, ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO 1999 – http://main3.amu.edu.pl/~wiadmat/145-151_zb_wm35.pdf

[2] Guła, L. W.: Several Treasures of the Queen of Mathematics – http://www.ijetae.com/files/Volume6Issue1/IJETAE_0116_09.pdf

[3] Guła, L. W.: The Truly Marvellous Proof – http://www.ijetae.com/files/Volume2Issue12/IJETAE_1212_14.pdf

[4] http://www.ams.org/profession/prizes-awards/ams-supported/beal-prize