The Truly Marvellous Proofs

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Abstract—

The truly marvellous proof of the Fermat's Last Theorem (FLT).
Three truly marvellous proofs of the Jeśmanowicz's Conjecture (JC).

The truly marvellous proof of the Beal's Conjecture in the case 2 (BC).

MSC: Primary: 11A41, 11D41, 11D45; Secondary: 11D61, 11D75, 11D85.

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I. INTRODUTION

The Fermat's Last Theorem is the famous theorem. The proof of FLT is dated July/August 1997.
The Jeśmanowicz's Conjecture [1] concerns a pythagorean triples, that is - the Diophantus Equation.
The Beal's Conjecture is a generalization of Fermat's Last Theorem. [5]

II. THE TRULY MARVELLOUS PROOF OF THE FERMAT'S LAST THEOREM

Theorem 1 (FLT). *For all* $n \in \{3,4,5,\dots\}$ *and for all* $A,B,C \in \{1,2,3,\dots\}$ *the equation*

$$A^n + B^n = C^n$$

has no primitive solutions $[A,B,C]$ *in* $\{1,2,3,\dots\}$.

Proof. Suppose that for some $n \in \{3,4,5,\dots\}$ and for some co-prime $A,B,C \in \{1,2,3,\dots\}$:

$$(A^n + B^n = C^n \wedge A + B > C \wedge A^2 + B^2 > C^2 \wedge \dots \wedge A^{n-1} + B^{n-1} > C^{n-1}).$$

In the another case we will have – For some $n \in \{3,4,5,\dots\}$ and for some co-prime $A,B,C \in \{1,2,3,\dots\}$:

$$(A^n + B^n = C^n \wedge A + B \le C \wedge A^2 + B^2 \le C^2 \wedge \dots \wedge A^{n-1} + B^{n-1} \le C^{n-1}) \Rightarrow$$

$$A^n + B^n < C^n,$$

which is inconsistent with $A^n + B^n = C^n$. [2]

Then only one number out of a given solution $[A,B,C]$ is even and the number $A + B - C$ is even. Without loss for this proof we can assume that $A, C - B \in \{1,3,5,\dots\}$ and that $A + B - C = 2v(u - v)$, inasmuch as – For all $u,v \in \{0,1,2,\dots\}$ such that $\boldsymbol{gcd}(u,v) = 1$ and $u - v \in \{\dots,-5,-3,-1,1,3,5,\dots\}$:

$$2v(u - v) \in \{\dots,-4,-2,0,2,4,\dots\}.$$

From the above it follows that – For each solution $[A,B,C]$ and for all relatively prime $u,v \in \{0,1,2,\dots\}$ such that $u - v \in \{\dots,-5,-3,-1,1,3,5,\dots\}$:

$$\neg[A = u^2 - v^2 \wedge B = 2uv \wedge -C = -(u^2 + v^2)] \Rightarrow$$

$$\{\neg[A = u^2 - v^2 \vee B = 2uv \vee -C = -(u^2 + v^2)] \wedge \neg[A + B - C = 2v(u - v)]\} \Rightarrow$$

$$2v(u - v) \notin \{\dots,-4,-2,0,2,4,\dots\},$$

which is inconsistent with $2v(u - v) \in \{\dots,-4,-2,0,2,4,\dots\}$.

This is the truly marvellous proof of FLT.

Theorem 2. *For all* $x,u,v \in \{1,2,3,\dots\}$ *such that* $\boldsymbol{gcd}(u,v) = 1$ *and* $u - v \in \{1,3,5,\dots\}$:

$$\left\{(u+v)^x(u-v)^x = \left[\frac{(u+v)^x + (u-v)^x}{2}\right]^2 - \left[\frac{(u+v)^x - (u-v)^x}{2}\right]^2 \wedge (u^2 - v^2)^{2+x} + (2uv)^{2+x} \right.$$
$$= (u^2 - v^2)^2(u^2 - v^2)^x + (2uv)^2(2uv)^x < (u^2 - v^2)^2(u^2 + v^2)^x + (2uv)^2(u^2 + v^2)^x$$
$$\left. = (u^2 + v^2)^{2+x}\right\}.$$

Theorem 3. *Let u and v be two relatively prime natural numbers such that $u-v$ is positive and odd. Then $(u^2-v^2, 2uv, u^2+v^2)$ is a primitive Pythagorean triple, and each primitive Pythagorean triple arises in this way for some u, v* [3], *that is to say for all primitive Pythagorean triple there exists different and only one shared pair (u, v).* [2]

III. THREE TRULY MARVELLOUS PROOFS OF THE JEŚMANOWICZ'S CONJECTURE

Conjecture 1 (Jeśmanowicz Conjecture). For all $x, y, z, u, v \in \{1,2,3, \dots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$:

$$(u^2-v^2)^x + (2uv)^y \neq (u^2+v^2)^z.$$

Proof 1. Suppose that for some $x, y, z, u, v \in \{1,2,3, \dots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$:

$$(u^2-v^2)^x + (2uv)^y = (u^2+v^2)^z.$$

On the strength of the Theorem 2 – For some $x, z, y, u, v \in \{1,2,3, \dots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$:

$$(u^2-v^2)^x = \left[\frac{(u+v)^x + (u-v)^x}{2}\right]^2 - \left[\frac{(u+v)^x - (u-v)^x}{2}\right]^2 =$$

$$\left[(u^2+v^2)^{\frac{z}{2}}\right]^2 - \left[(2uv)^{\frac{y}{2}}\right]^2 = \left[(u^2+v^2)^{\frac{z}{2}} + (2uv)^{\frac{y}{2}}\right]\left[(u^2+v^2)^{\frac{z}{2}} - (2uv)^{\frac{y}{2}}\right] \Rightarrow$$

$$\{[(u+v)^x + (u-v)^x]^2 = 2^2(u^2+v^2)^z \wedge [(u+v)^x - (u-v)^x]^2 = 2^2(2uv)^y\} \Rightarrow$$

$$z, y \in \{2,4,6, \dots\}.$$

At present we assume that the number $u^2 - v^2$ is minimal. [4]

Therefore – For some $x, Z, Y, u, v \in \{1,2,3, \dots\}$ and for some $z, y \in \{2,4,6, \dots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$:

$$(u-v)^x = (u^2+v^2)^Z - (2uv)^Y = (u^2+v^2)^{\frac{z}{2}} - (2uv)^{\frac{y}{2}} \Rightarrow u - v < u^2 - v^2,$$

which is inconsistent with minimal $u^2 - v^2$.

This is the truly marvellous proof 1 of JC.

Proof 2. Suppose that for some $x, y, z, u, v \in \{1,2,3, \dots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5, \dots\}$:

$$(u^2-v^2)^x + (2uv)^y = (u^2+v^2)^z.$$

If $u = 2$ and $v = 1$, then

$$(3^1 + 4^1 < 5^2 \wedge 3^1 + 4^2 < 5^2 \wedge 3^2 + 4^1 < 5^2 \wedge 3^3 + 4^2 < 5^3 \wedge 3^2 + 4^3 < 5^3 \wedge 3^3 + 4^3 < 5^3) \wedge$$
$$(3^1 + 4^1 > 5^1 \wedge 3^1 + 4^3 > 5^2 \wedge 3^3 + 4^1 > 5^2).$$

If $u - v > v$, then

$$(u^2-v^2)^1 + (2uv)^1 > (u^2+v^2)^1 \wedge$$
$$[(u^2-v^2)^1 + (2uv)^2 < (u^2+v^2)^2 \wedge (u^2-v^2)^2 + (2uv)^1 < (u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^1 + (2uv)^3 > (u^2+v^2)^2 \wedge (u^2-v^2)^3 + (2uv)^1 > (u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^3 + (2uv)^2 < (u^2+v^2)^3 \wedge (u^2-v^2)^2 + (2uv)^3 < (u^2+v^2)^3].$$

If $u - v < v$, then

$$(u^2-v^2)^1 + (2uv)^1 > (u^2+v^2)^1 \wedge$$
$$[(u^2-v^2)^1 + (2uv)^2 < (u^2+v^2)^2 \wedge (u^2-v^2)^2 + (2uv)^1 < (u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^1 + (2uv)^3 > (u^2+v^2)^2 \wedge (u^2-v^2)^3 + (2uv)^1 < (u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^3 + (2uv)^2 < (u^2+v^2)^3 \wedge (u^2-v^2)^2 + (2uv)^3 < (u^2+v^2)^3].$$

Moreover on the strength of the Theorem 2 – we will have: $(u^2-v^2)^3 + (2uv)^3 < (u^2+v^2)^3$.

Definition 1. $\boldsymbol{cpf}(\boldsymbol{p}u^2-\boldsymbol{p}v^2,\boldsymbol{p}2uv,\boldsymbol{p}u^2+\boldsymbol{p}v^2)=\boldsymbol{p}$, $\boldsymbol{p}$ is the odd common prime factor with the numbers of the solutions $[u^2-v^2,2uv,u^2+v^2]$ such that $\boldsymbol{p},u,v,u^2-v^2$ are co-prime. [2] This is the definition 1.

Therefore on the strength of the Theorem 2 – For some $z\in\{3,4,5,\ldots\}$ and for some $p,q\in\{0,1,2,\ldots\}$ and for some $u,v\in\{1,2,3,\ldots\}$ such that $p>q<z$ and $\boldsymbol{p},u,v,u^2-v^2$ are co-prime and $u-v\in\{1,3,5,\ldots\}$:

$$[(u^2-v^2)^{z+p}+(2uv)^{z-q}=(u^2+v^2)^z\vee(u^2-v^2)^{z-q}+(2uv)^{z+p}=(u^2+v^2)^z].$$

If $z+p>z\geq z-q$, then for some $\boldsymbol{p}=u^2+v^2$ we get:

$$[\boldsymbol{p}^{z+p}(u^2-v^2)^{z+p}+\boldsymbol{p}^{z-q}(2uv)^{z-q}=\boldsymbol{p}^z(u^2+v^2)^z=\boldsymbol{p}^{2z}]\Rightarrow$$

$$[\boldsymbol{p}^{p+q}(u^2-v^2)^{z+p}+(2uv)^{z-q}=\boldsymbol{p}^{z+q}\vee\boldsymbol{p}^p(u^2-v^2)^{z+p}+(2uv)^z=\boldsymbol{p}^z]\Rightarrow$$

$$\boldsymbol{gcd}(\boldsymbol{p},2uv)>1,$$

which is inconsistent with $\boldsymbol{p},u,v,u^2-v^2$ are co-prime.

Or – If $z+p>z\geq z-q$, then for some $\boldsymbol{p}$ we get:

$$[\boldsymbol{p}^{z-q}(u^2-v^2)^{z-q}+\boldsymbol{p}^{z+p}(2uv)^{z+p}=\boldsymbol{p}^z(u^2+v^2)^z=\boldsymbol{p}^{2z}]\Rightarrow$$

$$[(u^2-v^2)^{z-q}+\boldsymbol{p}^{p+q}(2uv)^{z+p}=\boldsymbol{p}^{z+q}\vee(u^2-v^2)^z+\boldsymbol{p}^p(2uv)^{z+p}=\boldsymbol{p}^z]\Rightarrow$$

$$\boldsymbol{gcd}(\boldsymbol{p},u^2-v^2)>1,$$

which is inconsistent with $\boldsymbol{p},u,v,u^2-v^2$ are co-prime.

This is the truly marvellous proof 2 of JC.

Proof 3. Suppose that for some $x,y,z,u,v\in\{1,2,3,\ldots\}$ such that $(x,y,z)\neq(2,2,2)$ and $\boldsymbol{gcd}(u,v)=1$ and $u-v\in\{1,3,5,\ldots\}$:

$$(u^2-v^2)^x+(2uv)^y=(u^2+v^2)^z.$$

On the strength of the Theorem 3 – For some $x,y,z,u,v\in\{1,2,3,\ldots\}$ such that $(x,y,z)\neq(2,2,2)$ and $\boldsymbol{gcd}(u,v)=1$ and $u-v\in\{1,3,5,\ldots\}$:

$$[(u^2-v^2)^x+(2uv)^y=(u^2+v^2)^z\wedge u^2-v^2+2uv>u^2+v^2\wedge(u^2-v^2)^2+(2uv)^2=(u^2+v^2)^2]\Rightarrow[(u^2-v^2)^x=(u^2-v^2)^2\wedge(2uv)^y=(2uv)^2\wedge(u^2+v^2)^z=(u^2+v^2)^2]\Rightarrow(x,y,z)=(2,2,2),$$

which is inconsistent with $(x,y,z)\neq(2,2,2)$. [2]

This is the truly marvellous proof 3 of JC.

IV. The Truly Marvellous Proof Of The Beal's Conjecture

Conjecture 2 (Beal Conjecture in the case 2). For all $x,y,z\in\{3,4,5\ldots\}$ the equation

$$A^x+B^y=C^z$$

has no primitive solutions $[A,B,C]$ in $\{1,2,3,\ldots\}$.

Proof of the Main Conjecture. Let for some $x,y,z\in\{3,4,5,\ldots\}$ and for some $A,B,C\in\{1,2,3,\ldots\}$ such that A,B and C are co-prime:

$$A^x+B^y=C^z.$$

Then only one number out of a given solution $[A,B,C]$ is even and the number $A+B-C$ is even. Without loss for this proof we can assume that $A,C-B\in\{1,3,5,\ldots\}$ and that $A+B-C=2v(u-v)$, inasmuch as – For all $u,v\in\{0,1,2,\ldots\}$ such that $\boldsymbol{gcd}(u,v)=1$ and $u-v\in\{\ldots,-5,-3,-1,1,3,5,\ldots\}$:

$$2v(u-v)\in\{\ldots,-4,-2,0,2,4,\ldots\}.$$

Therefore on the strength of the above two proofs of the Jeśmanowicz's Conjecture – For each solution $[A,B,C]$ and for all $u,v\in\{0,1,2,\ldots\}$ such that $\boldsymbol{gcd}(u,v)=1$ and $u-v\in\{\ldots,-5,-3,-1,1,3,5,\ldots\}$:

$$(A\neq u^2-v^2\wedge B\neq 2uv\wedge C\neq u^2+v^2)\Rightarrow A+B-C\neq 2v(u-v)\Rightarrow 2v(u-v)\notin\{\ldots,-4,-2,0,2,4,\ldots\},$$

which is inconsistent with $2v(u-v)\in\{\ldots,-4,-2,0,2,4,\ldots\}$.

This is the truly marvellous proof of BC.

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