Four Wonderful Proofs

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Abstract—
The wonderful proof of Fermat's Last Theorem.
Two wonderful proofs of Jeśmanowicz's Conjecture.

The wonderful proof of Beal's Conjecture in the case 2.

MSC: Primary: 11A41, 11D41, 11D45; Secondary: 11D61, 11D75, 11D85.

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I. INTRODUCTION

Fermat's Last Theorem is the famous theorem. The proof of FLT is dated July/August 1997.
Jeśmanowicz's Conjecture [1] concerns a pythagorean triples, that is - the Diophantus Equation.
Beal's Conjecture is the generalization of Fermat's Last Theorem. [4]

II. THE WONDERFUL PROOF OF THE FERMAT'S LAST THEOREM

Theorem 1 (Femat Last Theorem). *For all* $n \in \{3,4,5,\ldots\}$ *and for all* $A, B, C \in \{1,2,3,\ldots\}$:

$$A^n + B^n \neq C^n.$$

Proof. Suppose that for some $n \in \{3,4,5,\ldots\}$ and for some $A, B, C \in \{1,2,3,\ldots\}$:

$$(A^n + B^n = C^n \wedge A + B > C \wedge A^2 + B^2 > C^2 \wedge \ldots \wedge A^{n-1} + B^{n-1} > C^{n-1}).$$

In the another case we will have – For some $n \in \{3,4,5,\ldots\}$ and for some $A, B, C \in \{1,2,3,\ldots\}$:

$$(A^n + B^n = C^n \wedge A + B \leq C \wedge A^2 + B^2 \leq C^2 \wedge \ldots \wedge A^{n-1} + B^{n-1} \leq C^{n-1}) \Rightarrow A^n + B^n < C^n,$$

which is inconsistent with $A^n + B^n = C^n$. [2]

We assume that A, B and C are co-prime. Then only one number out of the solutions $[A, B, C]$ is even.

Without loss for this proof we can assume that $A, C - B \in \{1,3,5,\ldots\}$. and that $4 \nmid even\, B$. [2] Thus –

For some $u, v \in \{1,2,3,\ldots\}$ such that $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5,\ldots\}$: $A + B - C = 2v(u - v)$.

If $A + B - C = 2v(u - v)$ and $A = (u - v)^n + 2v(u - v)$, then odd $n \nmid A$. [3]

Therefore – For some $A, B, C \in \{1,2,3,\ldots\}$ and for some $u, v \in \{1,2,3,\ldots\}$ such that $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5,\ldots\}$:

$$\neg[A = u^2 - v^2 \vee B = 2uv \vee -C = -(u^2 + v^2)] \Rightarrow A + B - C \neq 2v(u - v),$$

which is inconsistent with $A + B - C = 2v(u - v)$. This is the wonderful proof.

Theorem 2. *For all* $\sigma \in \{1,2,3,\ldots\}$ *and for all* $u, v \in \{1,2,3,\ldots\}$ *such that* $u - v \in \{1,3,5,\ldots\}$ *and* $\boldsymbol{gcd}(u, v) = 1$:

$$(u^2 - v^2)^{2+\sigma} + (2uv)^{2+\sigma} < (u^2 + v^2)^{2+\sigma} \Rightarrow$$

$$(u^2 - v^2)^2 (u^2 - v^2)^\sigma + (2uv)^2 (2uv)^\sigma < (u^2 - v^2)^2 (u^2 + v^2)^\sigma + (2uv)^2 (u^2 + v^2)^\sigma.$$

III. TWO WONDERFUL PROOFS OF THE JEŚMANOWICZ'S CONJECTURE

Conjecture 1 (Jeśmanowicz Conjecture). For all $x, y, z, u, v \in \{1,2,3,\ldots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5,\ldots\}$:

$$(u^2 - v^2)^x + (2uv)^y \neq (u^2 + v^2)^z.$$

Proof 1. Suppose that for some $x, y, z, u, v \in \{1,2,3,\ldots\}$ such that $(x, y, z) \neq (2,2,2)$ and $\boldsymbol{gcd}(u, v) = 1$ and $u - v \in \{1,3,5,\ldots\}$:

$$(u^2 - v^2)^x + (2uv)^y = (u^2 + v^2)^z.$$

If $u=2$ and $v=1$, then

$$(3^1+4^2<5^2 \wedge 3^2+4^1<5^2 \wedge 3^1+4^3>5^2 \wedge 3^3+4^1>5^2 \wedge 3^3+4^2<5^3 \wedge\ 3^2+4^3<5^3).$$

If $u-v>v$, then

$$[(u^2-v^2)^1+(2uv)^2<(u^2+v^2)^2 \wedge (u^2-v^2)^2+(2uv)^1<(u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^1+(2uv)^3>(u^2+v^2)^2 \wedge (u^2-v^2)^3+(2uv)^1>(u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^3+(2uv)^2<(u^2+v^2)^3 \wedge (u^2-v^2)^2+(2uv)^3<(u^2+v^2)^3].$$

If $u-v<v$, then

$$[(u^2-v^2)^1+(2uv)^2<(u^2+v^2)^2 \wedge (u^2-v^2)^2+(2uv)^1<(u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^1+(2uv)^3>(u^2+v^2)^2 \wedge (u^2-v^2)^3+(2uv)^1<(u^2+v^2)^2] \wedge$$
$$[(u^2-v^2)^3+(2uv)^2<(u^2+v^2)^3 \wedge (u^2-v^2)^2+(2uv)^3<(u^2+v^2)^3].$$

On the strength of the Theorem 2 – For some $z \in \{3,4,5,\ldots\}$ and for some $p,q \in \{0,1,2,\ldots\}$ and for some $u,v \in \{1,2,3,\ldots\}$ such that $p+q>0$ and $\boldsymbol{gcd}(u,v)=1$ and $u-v \in \{1,3,5,\ldots\}$:

$$[(u^2-v^2)^{z+p}+(2uv)^{z-q}=(u^2+v^2)^z \vee (u^2-v^2)^{z-p}+(2uv)^{z+q}=(u^2+v^2)^z] \Rightarrow$$
$$[(u^2-v^2)^{z+p}+(2uv)^{z-q}=(u^2-v^2)^{z-p}+(2uv)^{z+q} \vee (u^2-v^2)^{z-p}+(2uv)^{z+q} = (u^2-v^2)^{z+p}+(2uv)^{z-q}] \Rightarrow$$
$$[(u^2-v^2)^{z+p}-(u^2-v^2)^{z-p}=(2uv)^{z+q}-(2uv)^{z-q}] \Rightarrow$$
$$(u^2-v^2)^z[(u^2-v^2)^p-(u^2-v^2)^{-p}]=(2uv)^z[(2uv)^q-(2uv)^{-q}] \Rightarrow p+q=0,$$

which is inconsistent with $p+q>0$. This is the wonderful proof.

Definition 1. $\boldsymbol{cpf}(\boldsymbol{p}u^2-\boldsymbol{p}v^2,\boldsymbol{p}2uv,\boldsymbol{p}u^2+\boldsymbol{p}v^2)=\boldsymbol{p}$, where is the odd common prime factor with the solution $[u^2-v^2,2uv,u^2+v^2]$.

Proof 2. For some $z \in \{3,4,5,\ldots\}$ and for some $p,q \in \{0,1,2,\ldots\}$ and for some $u,v \in \{1,2,3,\ldots\}$ such that $p+q>0$ and $\boldsymbol{gcd}(u,v)=1$ and $u-v \in \{1,3,5,\ldots\}$ and $\boldsymbol{gcd}(u^2-v^2,2uv,u^2+v^2)=1$:

$$[(u^2-v^2)^{z+p}+(2uv)^{z-q}=(u^2+v^2)^z \vee (u^2-v^2)^{z-p}+(2uv)^{z+q}=(u^2+v^2)^z].$$

If $z+p>z\geq z-q$, then for some $\boldsymbol{p}$ we get:

$$[\boldsymbol{p}^{z+p}(u^2-v^2)^{z+p}+\boldsymbol{p}^{z-q}(2uv)^{z-q}=\boldsymbol{p}^z(u^2+v^2)^z \wedge \boldsymbol{cpf}(\boldsymbol{p},\boldsymbol{p}u^2+\boldsymbol{p}v^2)=\boldsymbol{p}] \Rightarrow$$
$$\{[\boldsymbol{p}^{p+q}(u^2-v^2)^{z+p}+(2uv)^{z-q}=\boldsymbol{p}^q(u^2+v^2)^z \vee \boldsymbol{p}^p(u^2-v^2)^{z+p}+(2uv)^z=(u^2+v^2)^z] \wedge \boldsymbol{cpf}(\boldsymbol{p},\boldsymbol{p}u^2+\boldsymbol{p}v^2)=\boldsymbol{p}\} \Rightarrow [\boldsymbol{cpf}(\boldsymbol{p},(2uv)^{z-q})>1 \vee \boldsymbol{cpf}(\boldsymbol{p},(2uv)^z)>1],$$

which is inconsistent with $\boldsymbol{gcd}(u^2-v^2,2uv,u^2+v^2)=1$.

If $z+q>z\geq z-p$, then for some $\boldsymbol{p}$ we get:

$$[\boldsymbol{p}^{z-p}(u^2-v^2)^{z-p}+\boldsymbol{p}^{z+q}(2uv)^{z+q}=\boldsymbol{p}^z(u^2+v^2)^z \wedge \boldsymbol{cpf}(\boldsymbol{p},\boldsymbol{p}u^2+\boldsymbol{p}v^2)=\boldsymbol{p}] \Rightarrow$$
$$\{[(u^2-v^2)^{z-p}+\boldsymbol{p}^{q+p}(2uv)^{z+q}=\boldsymbol{p}^p(u^2+v^2)^z \vee (u^2-v^2)^z+\boldsymbol{p}^q(2uv)^{z+q}=(u^2+v^2)^z] \wedge \boldsymbol{cpf}(\boldsymbol{p},\boldsymbol{p}u^2+\boldsymbol{p}v^2)=\boldsymbol{p}\} \Rightarrow [\boldsymbol{cpf}(\boldsymbol{p},(u^2-v^2)^{z-p})>1 \vee \boldsymbol{cpf}(\boldsymbol{p},(u^2-v^2)^z)>1],$$

which is inconsistent with $\boldsymbol{gcd}(u^2-v^2,2uv,u^2+v^2)=1$. This is the wonderful proof.

IV. The Wonderful Proof Of The Beal's Conjecture

Conjecture 2 (Beal Conjecture in the case 2). For all $x,y,z \in \{3,4,5\ldots\}$ the equation

$$A^x+B^y=C^z$$

has no primitive solutions $[A,B,C]$ in $\{1,2,3,\ldots\}$.

Proof. Suppose that for some $x,y,z \in \{3,4,5,\ldots\}$ the equation

$$A^x+B^y=C^z$$

has primitive solutions $[A,B,C]$ in $\{1,2,3,\ldots\}$.

Then only one number out of the solutions $[A,B,C]$ is even.

Without loss for this proof we can assume that $A, C - B \in \{1,3,5,\ldots\}$. Thus – For some $u, v \in \{1,2,3,\ldots\}$ such that $\boldsymbol{gcd}(u,v) = 1$ and $u - v \in \{\ldots,-5,-3,-1,1,3,5,\ldots\}$: $A + B - C = 2v(u - v)$.

Hence on the strength of the above two proofs of Jeśmanowicz's Conjecture – For some $A, B, C \in \{1,2,3,\ldots\}$ and for some $u, v \in \{1,2,3,\ldots\}$ such that $\boldsymbol{gcd}(u,v) = 1$ and $u - v \in \{\ldots,-5,-3,-1,1,3,5,\ldots\}$:

$$\neg[A = u^2 - v^2 \vee B = 2uv \vee -C = -(u^2 + v^2)] \Rightarrow A + B - C \neq 2v(u - v),$$

which is inconsistent with $A + B - C = 2v(u - v)$. This is the wonderful proof.

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