SEVERAL TREASURES OF THE QUEEN OF MATHEMATICS

By Leszek W. Guła

Lublin-POLAND

lwgula@wp.pl

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Abstract

The Guła's Theorem (the new method of solving the equation $g = z^2 - y^2$ with given g).

The proper proof of The Fermat's Last Theorem (FLT)

Two complete proofs of The Beal's Conjecture.

The proof of The Erdős-Straus Conjecture.

The proof of The Jeśmanowicz's Conjecture.

Disproof The Oesterlé–Masser Conjecture (The ABC Conjecture).

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Keywords

ABC Conjecture, Algebra of Sets, Diophantine Equations, Diophantine Inequalities, Exponential Equations, Fermat Equation, Greatest Common Divisor, Newton Binomial Formula, Trinomial Square.

Dedicatory

I Dedicate this to My Wife

I. INTRODUCTION

The Guła's Theorem it is dated 03 and 04 June 1997 and concerns the Diophantine Equation.

The cover of this issue of the Bulletin is the frontispiece to a volume of Samuel de Fermat's 1670 edition of Bachet's Latin translation of Diophantus's Arithmetica. This edition includes the marginalia of the editor's father, Pierre de Fermat. Among these notes one finds the elder Fermat's extraordinary comment in connection with the Pythagorean equation $x^2 + y^2 = z^2$ the marginal comment that hints at the existence of a proof (a demonstratio sane mirabilis) of what has come to be known as Fermat's Last Theorem. Diophantus's work had fired the imagination of the Italian Renaissance mathematician Rafael Bombelli, as it inspired Fermat a century later. [3]

Around 1637, Fermat wrote his Last Theorem in the margin of his copy of the Arithmetica next to Diophantus sum-of-squares problem: it is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain. In number theory, Fermat's Last Theorem states that no three positive integers $A, B,$ and C can satisfy the equation $A^n + B^n = C^n$ for any integer value of n greater than two. [12]

Problem II.8 of the Diophantus's Arithmetica asks how a given square number is split into two other squares. Diophantus's shows how to solve this sum-of-squares problem for $k = 4$ and $u = 2$ [12], inasmuch as for all $k, u \in \{... - 2, -1,0,1,2, ...\}$:

$$k^2 = \left(\frac{2ku}{u^2+1}\right)^2 + \left[\frac{k(u^2-1)}{u^2+1}\right]^2. \quad [6]$$

Thus for all relatively prime natural numbers u, v such that $u - v \in \{1,3,5, ...\}$:

$$(u^2 + v^2)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 = (u^2 - v^2)^2 + (2uv)^2.$$

We have a primitive Pythagorean triple $(u^2 - v^2, 2uv, u^2 + v^2) = (x, y, z)$ the primitive triple because the numbers $x, y,$ and z are co-prime – pairwise relatively prime.

It is known that for some co-prime $x, y, z \in \{3,4,5, ...\}$:

$$[x^2 + y^2 = z^2 \land (x + y)^2 + (x - y)^2 = 2z^2], \qquad (1)$$

where z is odd because for all $a, b \in \{0,1,2,3, ...\}$: the number $\frac{(2a+1)^2+(2b+1)^2}{2}$ is odd.

It is easy to see that if $A^n + B^n = C^n$ then either $A, B,$ and C are co-prime or, if not co-prime that any common factor could be divided out of each term until the equation existed with co-prime bases. (Co-prime is synonymous with pairwise relatively prime and means that in a given set of numbers, no two of the numbers share a common factor). [10]

You could then restate FLT by saying that $A^n + B^n = C^n$ is impossible with co-prime bases. (Yes, it is also impossible without co-prime bases, but non co-prime bases can only exist as a consequence of co-prime bases). [10]

We assume that $\boldsymbol{gcd}(A, B) = \boldsymbol{gcd}(A, C) = \boldsymbol{gcd}(B, C) = 1$, otherwise we would divide each of three powers A^n, B^n, C^n by a power $\boldsymbol{gcd}(A, B, C)^n$ and the number $\boldsymbol{gcd}(A, B, C) > 1$ would be the greatest common divisor of these three numbers A, B, C.

Beal has formulated a conjecture in number theory on which he has been working for several years. The Beal Conjecture. Let $A, B, C, x, y,$ and z be positive integers with $x; y; z > 2$. If $A^x + B^y = C^z$ then $A, B,$ and C have a common factor [2] - with $\boldsymbol{gcd}(x, y, z) = 1$. Or, slightly restated also.

The Erdős-Straus Conjecture concerns the Diophantine Equations. One important topic in number theory is the study of Diophantine equations, equations in which only integer solutions are permitted. The type of Diophantine equation discussed in this paper concerns Egyptian fractions, which deal with the representation of rational numbers as the sum of three unit fractions. [4]

The Jeśmanowicz's Conjecture [1] concerns the Diophantine Equation.

The ABC Conjecture concerns easy equation $a + b = c$.

II. THE GUŁA'S THEOREM

Theorem 1 (Guła Theorem). *For each* $g \in \{8,12,16, ...\}$ *or for each* $g \in \{3,5,7, ...\}$ *there exist finitely many pairs* (z, y) *of positive integers such that*:

$$g = \left(\frac{g+d^2}{2d}\right)^2 - \left(\frac{g-d^2}{2d}\right)^2 = z^2 - y^2 = (z+y)(z-y) = \frac{g}{d}(z-y) = \frac{g}{d}d = g,$$

where $d|g$ *and* $d < \sqrt{g}$ *and -* $d, \frac{g}{d} \in \{2,4,6,\dots\}$ *with even* g *or* $d \in \{1,3,5,\dots\}$ *with odd* g.

Proof of The Main Theorem. For each $g \in \{8,12,16,\dots\}$ or for each $g \in \{3,5,7,\dots\}$ there exist finitely many of divisors d of fixed g such that $d < \sqrt{g}$ and $d, \frac{g}{d} \in \{2,4,6,\dots\}$ with even g, or $d \in \{1,3,5,\dots\}$ with odd g:

$$\left(\frac{g+d^2}{2d} = z \ \wedge \ \frac{g-d^2}{2d} = y = z - d\right).$$

The number of the pairs (z,y) is finite because $d|g$ with fixed g. This is the proof.

Theorem 2. *Let* u *and* v *be two relatively prime natural numbers such that* $u - v$ *is positive and odd. Then* $(u^2 - v^2, 2uv, u^2 + v^2)$ *is a primitive Pythagorean triple, and each primitive Pythagorean triple arises in this way for some* u, v [8], *that is to say for all primitive Pythagorean triple there exists different and only one shared pair* (u,v).

III. THE PROPER PROOF OF THE FERMAT'S LAST THEOREM

Theorem 3 (Femat Last Theorem). *For all* $n \in \{3,4,5,\dots\}$ *and for all* $A, B, C \in \{1,2,3,\dots\}$:

$$A^n + B^n \neq C^n.$$

Proof. Suppose that for some $n \in \{3,4,5,\dots\}$ the equation

$$A^n + B^n = C^n$$

has primitive solutions $[A,B,C]$ in $\{1,2,3,\dots\}$.

Then only one number out of (A,B,C) is even. Hence without loss of this proof we can assume that the numbers $A, C - B$ are odd.

Then also we will have - for some $n \in \{3,4,5,\dots\}$ and for some co-prime $A, B, C \in \{1,2,3,\dots\}$:

$$(A^n + B^n = C^n \ \wedge \ A + B > C \ \wedge \ A^2 + B^2 > C^2 \ \wedge \ \dots \ \wedge \ A^{n-1} + B^{n-1} > C^{n-1}),$$

otherwise $A^n + B^n < C^n$.

Every even number which is not the power of the number 2 has odd prime divisor, hence sufficient that we prove FLT for $n = 4$ and for odd prime numbers $n \in \mathbb{P}$. [5].

Without loss for this proof we can assume that $A, C - B \in \{1,3,5,\dots\}$ and that $4 \nmid B, C$.

A. Proof For $n = 4$.

For some $B \in \{6,10,14,\dots\}$ and for some $C, A, v \in \{1,3,5,\dots\}$, where A, B, C are co-prime:

$$\{[[B-(C-A)=2v\ \wedge\ (C-A+2v)^4=B^4=(C-A+A)^4-A^4]\ \wedge$$

$$[A-(C-B)=2v\ \wedge\ (C-B+2v)^4=A^4=(C-B+B)^4-B^4]]\Longrightarrow$$

$$[(C-A)^2 2v+\frac{3}{2}(C-A)(2v)^2+(2v)^3+\frac{4v^4}{C-A}=(C-A)^2A+\frac{3}{2}(C-A)A^2+A^3\ \wedge$$

$$(C-B)^2 2v+\frac{3}{2}(C-B)(2v)^2+(2v)^3+\frac{4v^4}{C-B}=(C-B)^2B+\frac{3}{2}(C-B)B^2+B^3]\}.$$

We assume that for some co-prime $c,d,e\in\{1,3,5,\dots\}$: $cde=v$, inasmuch as

$$A^2+B^2>C^2\Longrightarrow(2v)^2>2(C-A)(C-B).$$

From the above it follows that

$$(c^4+2cde=A\ \wedge\ 4d^4+2cde=B\ \wedge\ 4d^4=C-A\ \wedge\ c^4=C-B\ \wedge\ c^4+2cde=A).$$

Therefore for some co-prime $c,d,e\in\{1,3,5,\dots\}$:

$$(2cde+4d^4)^4=[(4d^4+A)^2]^2-(A^2)^2=\ [(4d^4+A)^2+A^2](2d^4+A)8d^4\Longrightarrow$$

$$2(ce+2d^3)^4=[(4d^4+A)^2]^2-(A^2)^2=\ [(4d^4+A)^2+A^2](2d^4+A).$$

Thus we assume that for some co-prime $z,w,x\in\{1,3,5,\dots\}$ and for some $y\in\{6,10,14,\dots\}$:

$$[zw=ce+2d^3\ \wedge\ x+y=2d^4+A+2d^4\ \wedge\ x=2d^4+A\ \wedge\ y=2d^4\ \wedge$$

$$4\nmid y\ \wedge\ 2(zw)^4=((x+y)^2+(x-y)^2)x=2(x^2+y^2)x\ \wedge$$

$$z^4w^4=(x^2+y^2)x\ \wedge\ (z^2)^2=(x^2+y^2)\ \wedge\ w^4=x]\Longrightarrow 4|y,$$

which is inconsistent with $4\nmid y$. This is the proper proof for $n=4$.

Remark 1. This assumption $(A^2)^2+(B^2)^2=(C^2)^2$ is false because $B\notin\{6,10,14,\dots\}$ in view of $2uv(u^2+v^2)\sqrt{2}=B$. On the strength of the Theorem 2: $(u^2+v^2)^2-(2uv)^2=(u^2-v^2)^2=A^2$ and $(u^2+v^2)^2+(2uv)^2=C^2\equiv 0$, which is obviously. Moreover on the strength of the Theorem 1 $[u^2+v^2=(uv)^2-1\ \vee\ u^2+v^2=u^2-v^2]\equiv 0$. This is the remark 1.

B. **Proof For** $n\in\mathbb{P}$. General Deductions.

Without loss for this proof we can assmue that $n\mid A$ in view of [7].

For some $n\in\mathbb{P}$ and some co-prime $A,C,B\in\{1,2,3,\dots\}$ such that $A,C-B\in\{1,3,5,\dots\}$:

$$[A-(C-B)=B-(C-A)=2v\ \wedge\ C-B+2v=A\ \wedge\ C-A+2v=B\ \wedge$$

$$(C-B+2v)^n=(C-B+B)^n-B^n\ \wedge\ (C-A+2v)^n=(C-A+A)^n-A^n\ \wedge$$

$$(A+B-B)^n+B^n=(A+B-2v)^n=C^n\ ].$$

Hence for some co-prime $C, B, A \in \{1,2,3, \ldots\}$ such that $C-B, A \in \{1,3,5, \ldots\}$:

$$\{(C-B)^{n-2}\nu+(n-1)(C-B)^{n-3}\,\nu^2+\cdots+2^{n-2}\nu^{n-1}+\frac{2^{n-1}\nu^n}{n(C-B)}=$$

$$\frac{B}{2}\left[(C-B)^{n-2}+\frac{n-1}{2}(C-B)^{n-3}B+\cdots+B^{n-2}\right]\ \wedge$$

$$(C-A)^{n-2}2\nu+\frac{n-1}{2}(C-A)^{n-3}\,(2\nu)^2+\cdots+(2\nu)^{n-1}+\frac{(2\nu)^n}{n(C-A)}=$$

$$A\left[(C-A)^{n-2}+\frac{n-1}{2}(C-A)^{n-3}A+\cdots+A^{n-2}\right]\ \wedge$$

$$(A+B)^{n-2}(-B)+\frac{n-1}{2}(A+B)^{n-3}\,(-B)^2+\cdots+(-B)^{n-1}=$$

$$(A+B)^{n-2}(-2\nu)+\frac{n-1}{2}(A+B)^{n-3}\,(-2\nu)^2+\cdots+(-2\nu)^{n-1}+\frac{(-2\nu)^n}{n(A+B)}\}.\ [7]$$

Thus for some $n \in \mathbb{P}$ and for some co-prime $e, m, c, h \in \{1,3,5, \ldots\}$:

$$[nemch=\nu\ \wedge\ n\nmid emch\ \wedge\ h^n=C-A\ \wedge\ n^{n-1}c^n=C-B.$$

B.1. Proof For Odd $A, B, C-B$.

For some $n \in \mathbb{P}$ and some co-prime $e, m\, c, h \in \{1,3,5, \ldots\}$ with $n \nmid emch$:

$$[n^{n-1}c^n+2nemch=A\ \wedge\ h^n+2nemch=B\ \wedge$$

$$2^n m^n=A+B=\ n^{n-1}c^n+h^n+4nemch\ \wedge\ n^{n-1}c^n+B=C]\Longrightarrow$$

$$[2^n m^n-h^n=\ n^{n-1}c^n+4nemch\ \wedge\ n\,|2m-h\ \wedge\ n^2\mid 2^n m^n-h^n]\Longrightarrow n\mid emch,$$

which is inconsistent with $n \nmid emch$. ♠

B.2. Proof For Even $B, C-A$.

For some $n \in \mathbb{P}$ and some co-prime $e, m, c, h \in \{1,3,5, \ldots\}$ with $n \nmid emch$:

$$[n^{n-1}c^n+2nemch=A\ \wedge\ 2^n h^n+2nemch=B\ \wedge$$

$$m^n=A+B=\ n^{n-1}c^n+2^n h^n+4nemch\ \wedge\ n^{n-1}c^n+B=C]\Longrightarrow$$

$$[m^n-2^n h^n=\ n^{n-1}c^n+4nemch\ \wedge\ n\mid m-2h\ \wedge\ n^2\mid\ m^n-2^n h^n]\Longrightarrow n\mid emch,$$

which is inconsistent with $n \nmid emch$. ♠ This is the proper proof for $n \in \mathbb{P}$. This is the proof.

IV. The Proof Of The Beal's Conjecture

Theorem 4. *For all* $x, y, z \in \{3,4,5, \dots\}$ *such that* x, y, z *are co-prime, the equation*

$$A^x + B^y = C^z$$

has no primitive solutions $[A, B, C]$ in $\{1,2,3, \dots\}$.

Proof. Suppose that for some $x, y, z \in \{3,4,5, \dots\}$ such that x, y, z are co-prime, the equation

$$A^x + B^y = C^z$$

has primitive solutions $[A, B, C]$ in $\{1,2,3, \dots\}$.

Let $\boldsymbol{p} \geq 2$ is the common factor of $(\boldsymbol{p}A), (\boldsymbol{p}B)$ and $(\boldsymbol{p}C)$.

If $(y > x \ \wedge \ y > z \ \wedge \ x \neq z)$, then

$$(\boldsymbol{p}^x, \boldsymbol{p}^y, \boldsymbol{p}^z) \Longrightarrow [(1, \boldsymbol{p}^{y-x}, \boldsymbol{p}^{z-x}) \ \vee \ (\boldsymbol{p}^{x-z}, \boldsymbol{p}^{y-z}, 1)] \ \Longrightarrow (\boldsymbol{p} \mid 1 \equiv 0).$$

If $(x < z \ \wedge \ y < z \ \wedge \ x \neq y)$, then

$$(\boldsymbol{p}^x, \boldsymbol{p}^y, \boldsymbol{p}^z) \Longrightarrow [(1, \boldsymbol{p}^{y-x}, \boldsymbol{p}^{z-x}) \ \vee \ (\boldsymbol{p}^{x-y}, 1, \boldsymbol{p}^{z-y})] \ \Longrightarrow (\boldsymbol{p} \mid 1 \equiv 0).$$

This is the proof.

Conjecture 1 (Beal Conjecture). For some $x, y, z \in \{3,4,5 \dots\}$ and for some $A, B, C \in \{1, 2,3, \dots\}$ such that $A, B,$ and C have the common factor $\boldsymbol{p}k \geq 2$:

$$A^x + B^y = C^z.$$

Or – For all $x, y, z \in \{3,4,5 \dots\}$ the equation

$$A^x + B^y = C^z$$

has no primitive solutions $[A, B, C]$ in $\{1,2,3, \dots\}$.

Proof 1. On the strength of the Theorem 4 and of the proof of FLT it must be

$$[y > x \geq z \ \wedge \ \boldsymbol{gcd}(y, x, z) = 1] \ \vee \ [x \leq y < z \ \wedge \ \boldsymbol{gcd}(x, y, z) = 1].$$

Lemma 1. For all $n \in \{3,4,5, \dots\}$ and for all $a, b \in \{1,2,3, \dots\}$ with $a > b$ and $\boldsymbol{gcd}(a, b) = 1$, and for some $c \in \{1, 2,3, \dots\}$ and for consecutive $k \in \{1, 2,3, \dots\}$:

$$\left[\frac{a^n - b^n}{a - b} = c \ \wedge \ \boldsymbol{gcd}(a, (a-b)c) = \boldsymbol{gcd}(b, (a-b)c) = 1 \wedge \left((a-b)c\right)^k = \boldsymbol{p}k\right] \Longrightarrow$$

$$\left[a\left((a-b)c\right)^k\right]^n - \left[b\left((a-b)c\right)^k\right]^n = \left((a-b)c\right)^{nk+1} \Longrightarrow A^n + B^{nk+1} = C^n.$$

This is the lemma 1.

Example 1. For all $n \in \{3,4,5,...\}$ and all $a \in \{2,3,4,...\}$ and for $b = 1$ and for $k = 2$ and for some $c \in \{1,2,3,...\}$ we will have

$$[(a^n - 1)^{2n} + (a^n - 1)^{2n+1} = (a(a^n - 1)^2)^n \ [11]\ ] \ \wedge \ (a^n - 1)^2 = \big((a-1)c\big)^k = \boldsymbol{p}k.$$

Naturally - For all $n \in \{3,4,5,...\}$ and for all $a \in \{2,3,4,...\}$ and for consecutive $k \in \{1,2,3,...\}$ there exist $x, y, z, A, B, C \in \{3,4,5,...\}$ such that $A, B,$ and C have the common factor $\boldsymbol{p}k \geq 7$:

$$[nk = x \ \wedge \ nk + 1 = y \ \wedge \ n = z \ \wedge \ a^n - 1 = A = B \ \wedge \ A^x + B^y = C^z].$$

This is the example 1.

Example 2. For $n = 3$ and for $a = 8$ and for $b = 7$ and for some $c = 13^2$ and for consecutive $k \in \{1,2,3,...\}$:

$$[8^3 - 7^3 = 13^2 \ \wedge \ (13^2)^k = \boldsymbol{p}k] \Longrightarrow$$

$$[7(13^2)^k]^3 + (13^2)^{3k+1} = [8(13^2)^k]^3 \Longrightarrow [7(13^2)^k]^3 + 13^{6k+2} = [8(13^2)^k]^3.$$

This is the example 2.

Lemma 2. For all $n \in \{3,5,7,...\}$ and for all $a, b \in \{1,2,3,...\}$ with $a > b$ and $\boldsymbol{gcd}(a,b) = 1$, and for some $c \in \{1,3,5,...\}$ and for consecutive $k \in \{1,2,3,...\}$:

$$\left[\frac{a^n + b^n}{a + b} = c \ [7] \ \wedge \ \boldsymbol{gcd}(a,(a+b)c) = \boldsymbol{gcd}(b,(a+b)c) = 1 \wedge \big((a+b)c\big)^k = \boldsymbol{p}k\right] \Longrightarrow$$

$$\left[a\big((a+b)c\big)^k\right]^n + \left[b\big((a+b)c\big)^k\right]^n = \big((a+b)c\big)^{nk+1} \Longrightarrow A^n + B^n = C^{nk+1}.$$

This is the lemma 2.

Lemma 3. For all $n \in \{4,6,8,...\}$ and for all $a \in \{1,3,5,...\}$ and for all $b \in \{2,4,6,...\}$ with $a > b$ and $\boldsymbol{gcd}(a,b) = 1$, and for some $c \in \{1,3,5,...\}$ and for consecutive $k \in \{1,2,3,...\}$:

$$[a^n + b^n = c \ \wedge \ \boldsymbol{gcd}(a,c) = \boldsymbol{gcd}(b,c) = 1 \wedge c^k = \boldsymbol{p}k] \Longrightarrow$$

$$(ac^k)^n + \ (bc^k)^n = c^{nk+1} \Longrightarrow \ A^n + B^n = C^{nk+1}.$$

This is the lemma 3.

Lemma 4. For all $n \in \{4,6,8,...\}$ and for all $a, b \in \{1,3,5,...\}$ with $a > b$ and $\boldsymbol{gcd}(a,b) = 1$, and for some $c \in \{1,3,5,...\}$ and for consecutive $k \in \{1,2,3,...\}$:

$$[a^n + b^n = 2c \ \wedge \ \boldsymbol{gcd}(a,c) = \boldsymbol{gcd}(b,c) = 1 \wedge (2c)^k = \boldsymbol{p}k] \Longrightarrow$$

$$[a(2c)^k]^n + [b(2c)^k]^n = (2c)^{nk+1} \Longrightarrow A^n + B^n = C^{nk+1}.$$

This is the lemma 4.

These lemmas gives all solutions of the equation $A^x + B^y = C^z$ such that $A, B,$ and C have the common factor $\boldsymbol{p}k \geq 2$, and the number of the solutions [A,B,C] is infinite. This is the proof 1.

Corollary 1. *For some* $x, y, z \in \{3,4,5 \dots\}$ *with* $[y > x \geq z \wedge \boldsymbol{gcd}(y, x, z) = 1]$ *or* $[x \leq y < z \wedge \boldsymbol{gcd}(x, y, z) = 1]$, *the equation*

$$A^x + B^y = C^z$$

has solutions $[A, B, C]$ *in* $\{1,2,3, \dots\}$ *such that* $A, B,$ *and* C *have the common factor* $\boldsymbol{p}k \geq 2$.

Proof 2. Suppose that for some $x, y, z \in \{3,4,5 \dots\}$ the equation

$$A^x + B^y = C^z$$

has primitive solutions $[A, B, C]$ in $\{1,2,3, \dots\}$.

On the strength of the Theorem 4 and of the Proof 1 of The Beal's Conjecture it must be –

For all $\boldsymbol{p} \in \{2,3,4, \dots\}$ and for some $n \in \{3,4,5, \dots\}$ and for some $k \in \{1, 2,3, \dots\}$ and for some $A, B, C \in \{1,2,3, \dots\}$ such that $A, B,$ and C are co-prime:

$$\boldsymbol{p}^n A^n + \boldsymbol{p}^{kn+1} B^{kn+1} = \boldsymbol{p}^n C^n.$$

Or – For all $\boldsymbol{p} \in \{2,3,4, \dots\}$ and for some $n \in \{3,4,5, \dots\}$ and for some $k \in \{1, 2,3, \dots\}$ and for some $A, B, C \in \{1,2,3, \dots\}$ such that $A, B,$ and C are co-prime:

$$\boldsymbol{p}^n A^n + \boldsymbol{p}^n B^n = \boldsymbol{p}^{kn+1} C^{kn+1}.$$

Hence

For any $\boldsymbol{p} \in \{2,3,4, \dots\}$ and for some $n \in \{3,4,5, \dots\}$ and for some $k \in \{1, 2,3, \dots\}$ and for some $A, B, C \in \{1,2,3, \dots\}$ such that $A, B,$ and C are co-prime:

$$\{A^n + \boldsymbol{p}^{kn-n+1} B^{kn+1} = C^n \;\wedge\; [\boldsymbol{gcd}(\boldsymbol{p}, A^n) > 1 \;\vee\; \boldsymbol{gcd}(\boldsymbol{p}, C^n) > 1]\}.$$

Or – For any $\boldsymbol{p} \in \{2,3,4, \dots\}$ and for some $n \in \{3,4,5, \dots\}$ and for some $k \in \{1, 2,3, \dots\}$ and for some $A, B, C \in \{1,2,3, \dots\}$ such that $A, B,$ and C are co-prime:

$$\{A^n + B^n = \boldsymbol{p}^{kn-n+1} C^{kn+1} \;\wedge\; [\boldsymbol{gcd}(\boldsymbol{p}, A^n) > 1 \;\vee\; \boldsymbol{gcd}(\boldsymbol{p}, B^n) > 1]\}.$$

Thus $\boldsymbol{gcd}(A, B, C) > 1$, which is inconsistent with $A, B,$ and C are co-prime. This is the proof 2.

Corollary 2. *For all* $x, y, z \in \{3,4,5 \dots\}$ *with* $[y > x \geq z \wedge \boldsymbol{gcd}(y, x, z) = 1]$ *or* $[x \leq y < z \wedge \boldsymbol{gcd}(x, y, z) = 1]$, *the equation*

$$A^x + B^y = C^z$$

has no primitive solutions $[A, B, C]$ *in* $\{1,2,3, \dots\}$.

V. THE PROOF OF THE ERDŐS-STRAUS CONJECTURE

Conjecture 2 (Erdős– Straus Conjecture). For all $n \in \{2,3,4,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Proof. For all $n \in \{2,4,6,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ \wedge \ \frac{n}{2}=a \ \wedge \ \frac{n+2}{2}=b \ \wedge \ \frac{n(n+2)}{4}=\mathrm{c}\right]. \ \spadesuit$$

For all $n \in \{3,7,11,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \wedge \frac{n+1}{2}=a=c \ \wedge \ \frac{n(n+1)}{4}=b\right]. \tag{2}$$

Thus for all $n \in \{3,9,15,21,27,33,39,45,51,57,63,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \wedge \frac{2n}{3}=a=c \ \wedge \ n=b\right],$$

and for all $n \in \{7,21,35,49,63,77,91,105,119,133,147,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \wedge \frac{4n}{7}=a=c \ \wedge \ 2n=b\right],$$

and for all $n \in \{11,33,55,77,99,121,143,165,187,209,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \wedge \frac{6n}{11}=a=c \ \wedge \ 3n=b\right],\dots.$$

For all $n \in \{5,13,21,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{3n+1}{4}=a \ \wedge \ \frac{n(3n+1)}{4}=b \ \wedge \ \frac{3n+1}{8}=c\right]. \tag{3}$$

For all $n \in \{5,11,17,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{n+1}{3}=a \ \wedge \ n=b \ \wedge \ \frac{n(n+1)}{3}=c\right]. \tag{4}$$

On the strength of [4] for all $n \in \{97,111,125,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{2(n+1)}{7}=a \ \wedge \ 2n=b \ \wedge \ \frac{2n(n+1)}{7}=c\right]. \tag{5}$$

For all $n \in \{3,17,31,\dots\}$ and for some $c,x \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{2n}+\frac{1}{c}+\frac{1}{c+x} \wedge \frac{4n^2-1}{7}=x \ \wedge \ \frac{2n+1}{7}=c\right]. \tag{6}$$

On the strength of $[4]$ for all $n \in \{13,33,53,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{3n+1}{10}=a \wedge \frac{n(3n+1)}{4}=b \wedge \frac{3n+1}{2}=c\right]. \qquad (7)$$

The key of this proof is the following sum of the subsets, which should be written in columns to determine the missing odd prime numbers, namely

$$\{5,9,13,\dots\}=$$

$$\{5,25,45,\dots\} \cup \{9,29,49,\dots\} \cup \{13,33,53,\dots\} \cup \{17,37,57,\dots\} \cup \{21,41,61,\dots\}.$$

From $(3),(4),(5),(6),\text{and }(7)$ we will have by analogy yes, as well from (2).

Therefore, and on the strength of the Theorem 1 and the Theorem 2 we get -

for all $n \in \{n\colon\ n=337+120k \wedge k \in [0,1,2,\dots] \wedge \mathit{only}\ 1,n \mid n\}$

or for all $n \in \{n\colon\ n=1009+120k \wedge k \in [0,1,2,\dots] \wedge \mathit{only}\ 1,n \mid n\}$

or for all $n \in \{n\colon\ n=1201+120k \wedge k \in [0,1,2,\dots] \wedge \mathit{only}\ 1,n \mid n\}$

or for some $m,x,c \in \{1,2,3,\dots\}$ and for some $d \in \{2,4,6,\dots\}$ such that $2nm > d$:

$$\left[\frac{4}{n}=\frac{1}{nm}+\frac{1}{x+c}+\frac{1}{c} \Rightarrow (4m-1)c^2+[(4m-1)x-2nm]c-nmx=0\right] \Rightarrow$$

$$\left[\Delta=[(4m-1)x]^2+(2nm)^2=(n^2+m^2)^2 \wedge x=\frac{n^2-m^2}{4m-1} \wedge c=\frac{nm+m^2}{4m-1}\right] \vee$$

$$\left[\Delta=[(4m-1)x]^2+(2nm)^2=\left(\frac{(2nm)^2+d^2}{2d}\right)^2 \wedge x=\frac{\frac{(2nm)^2}{2d}-\frac{d}{2}}{4m-1} \wedge c=\frac{nm+\frac{d}{2}}{4m-1}\right].$$

This is the proof.

Example 3. For $n=337$ and for $m=18$ and for some $x,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4m-1}{nm}=\frac{x+2c}{(x+c)c} \wedge 71c^2+(71x-12132)c-6066x=0\right] \Rightarrow$$

$$\left[\Delta=(71x)^2+12132^2=(n^2+m^2)^2 \wedge x=\frac{337^2-18^2}{71} \wedge c=\frac{6066+324}{71}=90\right].$$

Hence for all $n \in \{337,1011,1685,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{1685n}{337}=(x+c)=a \wedge 18n=b \wedge \frac{90n}{337}=c\right].$$

This is the example 3.

Example 4. For $n = 1009$ and for $m = 3$ and for some $x, c \in \{1,2,3, \dots\}$:

$$\left[\frac{4m-1}{nm} = \frac{x+2c}{(x+c)c} \ \wedge \ 3c^2 + (3x - 6054)c - 3027x = 0\right] \Longrightarrow$$

$$\left[\Delta = (3x)^2 + 6054^2 = (n^2+m^2)^2 \ \wedge \ x = \frac{1009^2 - 3^2}{11} \ \wedge \ c = \frac{3027+9}{11} = 276\right].$$

Hence for all $n \in \{1009,3027,5045, \dots\}$ and for some $a, b, c \in \{1,2,3, \dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \wedge \ \frac{92828n}{1009} = (x+c) = a \ \wedge \ 3n = b \ \wedge \ \frac{276n}{1009} = c\right].$$

This is the example 4.

Example 5. For $n = 1201$ and for $m = 8$ and for some $x, c \in \{1,2,3, \dots\}$:

$$\left[\frac{31}{8n} = \frac{x+2c}{(x+c)c} \ \wedge \ 31c^2 + (31x - 19216)c - 9608x = 0\right] \Longrightarrow$$

$$\left[\Delta = (31x)^2 + 19216^2 = (n^2+m^2)^2 \ \wedge \ x = \frac{1201^2 - 8^2}{31} \ \wedge \ c = \frac{nm+m^2}{31}\right].$$

Hence for all $n \in \{1201,3603,6005, \dots\}$ and for some $a, b, c \in \{1,2,3, \dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \wedge \ \frac{46839n}{1201} = a \ \wedge \ 8n = b \ \wedge \ \frac{312n}{1201} = c\right].$$

This is the example 5.

VI. The Proof Of The Jeśmanowicz's Conjecture

Conjecture 3 (Jesmanowicz Conjecture). For all $p, q \in \{0,1,2, \dots\}$ and for all $x, r, s \in \{1,2,3, \dots\}$ such that $p + q > 0$ and $r - s \in \{1,3,5, \dots\}$ and $\boldsymbol{gcd}(r,s) = 1$:

$$[(r^2 - s^2)^{p+x} + (2rs)^x \neq (r^2+s^2)^{q+x} \ \wedge \ (r^2 - s^2)^x + (2rs)^{p+x} \neq (r^2+s^2)^{q+x}].$$

Proof. Suppose that for some $p, q \in \{0,1,2, \dots\}$ and for some $x, r, s \in \{1,2,3, \dots\}$ such that $p + q > 0$ and $r - s \in \{1,3,5, \dots\}$ and $\boldsymbol{gcd}(r,s) = 1$:

$$[(r^2 - s^2)^{p+x} + (2rs)^x = (r^2+s^2)^{q+x} \vee (r^2 - s^2)^x + (2rs)^{p+x} = (r^2+s^2)^{q+x}].$$

On the strength of the Theorem 2 and of the Theorem 4 it must be

$$(p + x = 2 \ \wedge \ p = 0 \ \wedge \ x = 2 \ \wedge \ q + x = 2 \ \wedge \ q = 0) \ \Longrightarrow \ p + q = 0,$$

which is inconsistent with $p + q > 0$. This is the proof.

V. DISPROOF THE ABC CONJECTURE

Conjecture 4 (ABC Conjecture). For all $\epsilon > 0$ there exist only finitely many triples (a, b, c) of positive coprime integers, with $a + b = c > d^{1+\epsilon}$, where d denotes the product of the distinct prime factors of the product abc. [9]

Disproof. *For all* $n \in \{1,3,5, \dots\}$ *and for all relatively prime* $A, B \in \{1,2,3, \dots\}$ *there exists the natural number* $g > 0$ [7] *and there exists the number* $\epsilon = c$:

$$A^n + B^n = (A + B)\frac{A^n + B^n}{A + B} = (A + B)g = a + b = \epsilon = c < d^{1+\epsilon},$$

where d *denotes the product of the distinct prime factors of the product* abc. This is the disproof.

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