SEVERAL TREASURES OF THE QUEEN OF MATHEMATICS

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Abstract

The new method of solving the equation $g = z^2 - y^2$ in positive integers with given g.

The complete proof of The Erdős-Straus Conjecture.

The proof of The Jeśmanowicz's Conjecture.

The refutation of The Oesterlé–Masser Conjecture (The ABC Conjecture).

MSC: Primary - 11A41, 11D41, 11D45; Secondary - 11D61, 11D75, 11D85.

Keywords

ABC Conjecture, Algebra of Sets, Diophantine Equations, Diophantine Inequalities, Exponential Equations, Fermat Equation, Greatest Common Divisor, Prime Numbers, Pythagoras Equation, Trinomial Square.

Dedicatory

I Dedicate this to My Wife

I. INTRODUCTION

The new method of solving the equation $g = z^2 - y^2$ in positive integers with given g and The Erdős-Straus Conjecture and The Jeśmanowicz's Conjecture concerns the Diophantine Equations. One important topic in number theory is the study of Diophantine equations, equations in which only integer solutions are permitted. The type of Diophantine equation discussed in this paper concerns Egyptian fractions, which deal with the representation of rational numbers as the sum of three unit fractions. [3] The Jeśmanowicz Conjecture [2] is slightly restated because The Fermat's Last Theorem (FLT) is true. [4], [5] The ABC Conjecture concerns easy equation $a + b = c$.

II. THE MAIN THEOREM

Theorem 1. *For each given* $g \in \{8,12,16,\dots\}$ *or for each given* $g \in \{3,5,7,\dots\}$ *there exist finitely many pairs* (z, y) *of positive integers such that*:

$$g = \left(\frac{g + d^2}{2d}\right)^2 - \left(\frac{g - d^2}{2d}\right)^2 = z^2 - y^2 = (z + y)(z - y) = \frac{g}{d}(z - y) = \frac{g}{d}d = g,$$

where $d|g$ *and* $d < \sqrt{g}$ *and -* $d, \frac{g}{d} \in \{2,4,6,\dots\}$ *with even* g *or* $d \in \{1,3,5,\dots\}$ *with odd* g.

Proof of the Main Theorem. It is easy to verify that

$$\left(\frac{g+d^2}{2d}=z \ \wedge \ \frac{g-d^2}{2d}=y=z-d\right).$$

The number of the pairs (z,y) is finite because $d|g$ with given g. Moreover this pairs (z,y) of positive integers are all, inasmuch as FLT for even n is true. ♠

III. THE PROOF OF THE ERDŐS-STRAUS CONJECTURE

Conjecture 1 (Erdős–Straus Conjecture). For all $n \in \{2,3,4,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Proof. For all $n \in \{2,4,6,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ \wedge \ \frac{n}{2}=a \ \wedge \ \frac{n+2}{2}=b \ \wedge \ \frac{n(n+2)}{4}=\mathrm{c}\right]. \ ♠$$

For all $n \in \{3,7,11,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \wedge \frac{n+1}{2}=a=c \ \wedge \ \frac{n(n+1)}{4}=b\right]. \tag{1}$$

Thus for all $n \in \{3,9,15,21,27,33,39,45,51,57,63,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \ \wedge \ \frac{2n}{3}=a=c \ \wedge \ n=b\right],$$

and for all $n \in \{7,21,35,49,63,77,91,105,119,133,147,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \ \wedge \ \frac{4n}{7}=a=c \ \wedge \ 2n=b\right],$$

and for all $n \in \{11,33,55,77,99,121,143,165,187,209,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \ \wedge \ \frac{6n}{11}=a=c \ \wedge \ 3n=b\right],\dots.$$

For all $n \in \{5,13,21,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{3n+1}{4}=a \ \wedge \ \frac{n(3n+1)}{4}=b \ \wedge \ \frac{3n+1}{8}=c\right]. \tag{2}$$

For all $n \in \{5,11,17,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{n+1}{3}=a \ \wedge \ n=b \ \wedge \ \frac{n(n+1)}{3}=c\right]. \tag{3}$$

On the strength of $[3]$ for all $n \in \{97,111,125, \dots\}$ and for some $a, b, c \in \{1,2,3, \dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \wedge \frac{2(n+1)}{7} = a \wedge 2n = b \wedge \frac{2n(n+1)}{7} = c\right]. \qquad (4)$$

For all $n \in \{3,17,31, \dots\}$ and for some $c, x \in \{1,2,3, \dots\}$:

$$\left[\frac{4}{n} = \frac{1}{2n} + \frac{1}{c} + \frac{1}{c+x} \wedge \frac{4n^2-1}{7} = x \wedge \frac{2n+1}{7} = c\right]. \qquad (5)$$

On the strength of $[3]$ for all $n \in \{13,33,53, \dots\}$ and for some $a, b, c \in \{1,2,3, \dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \wedge \frac{3n+1}{10} = a \wedge \frac{n(3n+1)}{4} = b \wedge \frac{3n+1}{2} = c\right]. \qquad (6)$$

The key to the proof is the following sum of the subsets, which should be written in columns to determine the missing odd prime numbers

$$\{5,9,13, \dots\} = \{5,25,45, \dots\} \cup \{9,29,49, \dots\} \cup \{13,33,53, \dots\} \cup \{21,41,61, \dots\}.$$

From $(2), (3), (4), (5)$ and (6) we will have by analogy yes, as well from (1).

Therefore, and on the strength of the **Theorem 1** we get -

for all $n \in \{n\colon\ n = 337 + 120k \wedge k \in [0,1,2, \dots] \wedge only\ 1, n \mid n\}$

or for all $n \in \{n\colon\ n = 1009 + 120k \wedge k \in [0,1,2, \dots] \wedge only\ 1, n \mid n\}$

or for all $n \in \{n\colon\ n = 1201 + 120k \wedge k \in [0,1,2, \dots] \wedge only\ 1, n \mid n\}$

or for some $m, x, c \in \{1,2,3, \dots\}$ and for some $d \in \{2,4,6, \dots\}$ such that $2nm > d$:

$$\left[\frac{4}{n} = \frac{1}{nm} + \frac{1}{x+c} + \frac{1}{c} \Longrightarrow (4m-1)c^2 + [(4m-1)x - 2nm]c - nmx = 0\right] \Longrightarrow$$

$$\left[\Delta = [(4m-1)x]^2 + (2nm)^2 = (n^2+m^2)^2 \wedge x = \frac{n^2-m^2}{4m-1} \wedge c = \frac{nm+m^2}{4m-1}\right] \vee$$

$$\left[\Delta = [(4m-1)x]^2 + (2nm)^2 = \left(\frac{(2nm)^2+d^2}{2d}\right)^2 \wedge x = \frac{\frac{(2nm)^2}{2d} - \frac{d}{2}}{4m-1} \wedge c = \frac{nm + \frac{d}{2}}{4m-1}\right]. \ ♠$$

Examples.

For $n = 337$ and for $m = 18$ and for some $x, c \in \{1,2,3, \dots\}$:

$$\left[\frac{4m-1}{nm} = \frac{x+2c}{(x+c)c} \wedge 71c^2 + (71x - 12132)c - 6066x = 0\right] \Longrightarrow$$

$$\left[\Delta = (71x)^2 + 12132^2 = (n^2+m^2)^2 \;\wedge\; x = \frac{337^2 - 18^2}{71} \;\wedge\; c = \frac{6066+324}{71} = 90\right].$$

Hence for all $n \in \{337,1011,1685,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;\wedge\; \frac{1685n}{337} = (x+c) = a \;\wedge\; 18n = b \;\wedge\; \frac{90n}{337} = c\right].$$

For $n = 1009$ and for $m = 3$ and for some $x,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4m-1}{nm} = \frac{x+2c}{(x+c)c} \;\wedge\; 3c^2 + (3x-6054)c - 3027x = 0\right] \Longrightarrow$$

$$\left[\Delta = (3x)^2 + 6054^2 = (n^2+m^2)^2 \;\wedge\; x = \frac{1009^2 - 3^2}{11} \;\wedge\; c = \frac{3027+9}{11} = 276\right].$$

Hence for all $n \in \{1009,3027,5045,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;\wedge\; \frac{92828n}{1009} = (x+c) = a \;\wedge\; 3n = b \;\wedge\; \frac{276n}{1009} = c\right].$$

For $n = 1201$ and for $m = 8$ and for some $x,c \in \{1,2,3,\dots\}$:

$$\left[\frac{31}{8n} = \frac{x+2c}{(x+c)c} \;\wedge\; 31c^2 + (31x-19216)c - 9608x = 0\right] \Longrightarrow$$

$$\left[\Delta = (31x)^2 + 19216^2 = (n^2+m^2)^2 \;\wedge\; x = \frac{1201^2 - 8^2}{31} \;\wedge\; c = \frac{nm+m^2}{31}\right].$$

Hence for all $n \in \{1201,3603,6005,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;\wedge\; \frac{46839n}{1201} = a \;\wedge\; 8n = b \;\wedge\; \frac{312n}{1201} = c\right].$$

IV. THE PROOF OF THE JEŚMANOWICZ'S CONJECTURE

Conjecture 2 (Jeśmanowicz Conjecture). For all $p,q \in \{0,1,2,\dots\}$ and for all $x,r,s \in \{1,2,3,\dots\}$ such that $p+q > 0$ and $r-s \in \{1,3,5,\dots\}$ and $\mathbf{gcd}(r,s) = 1$:

$$[(r^2-s^2)^{p+x} + (2rs)^x \neq (r^2+s^2)^{q+x} \;\wedge\; (r^2-s^2)^x + (2rs)^{p+x} \neq (r^2+s^2)^{q+x}].$$

Lemma. *Let r and s be two relatively prime natural numbers such that $r-s$ is positive and odd. Then $(r^2-s^2, 2rs, r^2+s^2)$ is a primitive Pythagorean triple, and each primitive Pythagorean triple arises in this way for some r,s* [6]*, that is to say for all primitive Pythagorean triple there exists different and only one shared pair (r,s).*

Proof. Suppose that for some $p,q \in \{0,1,2,\dots\}$ and for some $x,r,s \in \{1,2,3,\dots\}$ such that $p+q > 0$ and $r-s \in \{1,3,5,\dots\}$ and $\mathbf{gcd}(r,s) = 1$:

$$[(r^2 - s^2)^{p+x} + (2rs)^x = (r^2 + s^2)^{q+x} \lor (r^2 - s^2)^x + (2rs)^{p+x} = (r^2 + s^2)^{q+x}].$$

On the strength of the above **Lemma** it must be

$$(p + x = 2 \land p = 0 \land x = 2 \land q + x = 2 \land q = 0) \Longrightarrow p + q = 0,$$

which is inconsistent with $p + q > 0$. ♠

V. THE REFUTATION OF THE ABC CONJECTURE

Conjecture 3 (ABC Conjecture). For all $\epsilon > 0$ there exist only finitely many triples (a, b, c) of positive coprime integers, with $a + b = c > d^{1+\epsilon}$, where d denotes the product of the distinct prime factors of the product abc. [1]

Refutation . Without loss for this disproof we can assume that $g = 1$ or g is odd prime number. **Yes, without loss for this refutation.**

Theorem 2. *Let* $a1$ *and* $b1$ *be two relatively prime natural numbers and* $x \in \{1,3,5, \ldots\}$. *If* $(a1)^x + (b1)^x = a + b = (a1 + b1)g = c > d^{1+1/c} = d^{1+\epsilon}$, *where* d *denotes the product of the distinct prime factors of the products* $a1b1(a1 + b1)g$ *and* abc, *then for each* $x \in \{1,3,5, \ldots\}$ *and for* $\epsilon = 1/c$ *and for each* $\epsilon \in (0, 1/c)$: $a + b = c > d^{1+\epsilon}$. *Therefore there exist infinitely many triples* (a, b, c) *of positive coprime integers, with* $a + b = c > d^{1+\epsilon}$, *where* d *denotes the product of the distinct prime factors of the product* abc. This is the refutation.

Examples.

For all $x \in \{1,3,5, \ldots\}$ and for $\epsilon = \frac{1}{5^x + 27^x}$:

$$5^x + 27^x = 2^5 g > (2 \cdot 3 \cdot 5g)^{1+\epsilon} = d^{1+\epsilon}.$$

For all $x \in \{1,3,5, \ldots\}$ and for $\epsilon = \frac{1}{1^x + 63^x}$:

$$1^x + 63^x = 2^6 g > (2 \cdot 3 \cdot 7g)^{1+\epsilon} = d^{1+\epsilon}.$$

For all $x \in \{1,3,5, \ldots\}$ and for $\epsilon = \frac{1}{3^x + 125^x}$:

$$\left[\left(\frac{3^x + 125^x}{3 + 125} \text{ is odd [4]}\right) \land 3^x + 125^x = 2^7 g > (2 \cdot 3 \cdot 5g)^{1+\epsilon} = d^{1+\epsilon}\right] \Longrightarrow$$

$$2^6 > (2g)^\epsilon (3 \cdot 5)^{1+\epsilon}.$$

For all $x \in \{1,3,5, \ldots\}$ and for $\epsilon = \frac{1}{81^x + 175^x}$:

$$81^x + 175^x = 2^8 g > (2 \cdot 3 \cdot 5 \cdot 7g)^{1+\epsilon} = d^{1+\epsilon}.$$

For all $x \in \{1,3,5, \ldots\}$ and for $\epsilon = \frac{1}{169^x + 343^x}$:

$$169^x + 343^x = 2^9 g > (2 \cdot 7 \cdot 13g)^{1+\epsilon} = d^{1+\epsilon}.$$

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