THE PROOF OF THE ERDŐS-STRAUS CONJECTURE AND THE PROOF OF THE JEŚMANOWICZ'S CONJECTURE

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Abstract

The complete proof of the Erdős-Straus Conjecture.

The proof of the Jeśmanowicz's Conjecture.

MSC: Primary - 11A41, 11D45; Secondary - 11D41, 11D61.

Keywords

Algebra of Sets, Diophantine Equations, Diophantine Inequalities, Exponential Equations, Prime Numbers, Pythagoras Equation, Trinomial Square.

Dedicatory

I Dedicate this to My Wife

I. INTRODUCTION

The Erdős-Straus Conjecture and the Jeśmanowicz's Conjecture concerns the Diophantine Equations. One important topic in number theory is the study of Diophantine equations, equations in which only integer solutions are permitted. The type of Diophantine equation discussed in this paper concerns Egyptian fractions, which deal with the representation of rational numbers as the sum of three unit fractions. [2] The Jeśmanowicz conjecture [1] is slightly restated because The Fermat Last Theorem is true. [3]

II. THE PROOF OF THE ERDŐS-STRAUS CONJECTURE (ESC)

Conjecture (Erdős– Straus Conjecture). For all $n \in \{2,3,4,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Proof. For all $n \in \{2,4,6,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ \wedge\ \frac{n}{2}=a \ \wedge\ \frac{n+2}{2}=b \ \wedge\ \frac{n(n+2)}{4}=\mathrm{c}\right].\ \spadesuit$$

For all $n \in \{3,7,11,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{c}+\frac{1}{b} \wedge \frac{n+1}{2}=a=c \ \wedge\ \frac{n(n+1)}{4}=b\right].\ \spadesuit \qquad (1)$$

It is easy to verify that

$$\{5,9,13,\dots\} = \{5,25,45,\dots\} \cup \{9,29,49,\dots\} \cup \{13,33,53,\dots\} \cup \{21,41,61,\dots\}.$$

On the strength of $[2]$ for all $n \in \{13,33,53,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \wedge\ \frac{3n+1}{10} = a \ \wedge\ \frac{n(3n+1)}{4} = b \ \wedge\ \frac{3n+1}{2} = c\right]. \ \spadesuit$$

From (1) we obtain:

for all $n \in \{3,9,15,21,27,33,39,45,51,57,63,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{c} + \frac{1}{b} \ \wedge\ \frac{2n}{3} = a = c \ \wedge\ n = b\right],$$

and for all $n \in \{7,21,35,49,63,77,91,105,119,133,147,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{c} + \frac{1}{b} \ \wedge\ \frac{4n}{7} = a = c \ \wedge\ 2n = b\right],$$

and for all $n \in \{11,33,55,77,99,121,143,165,187,209,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{c} + \frac{1}{b} \ \wedge\ \frac{6n}{11} = a = c \ \wedge\ 3n = b\right], \dots . \ \spadesuit$$

From the above we get

$$\{5,25,65,85,125,145,185,205,245,265,305,325,\dots\} \cup$$

$$\{29,89,109,149,169,229,269,289,349,389,\dots\} \cup$$

$$\{41,61,101,181,221,241,281,401,421,\dots\}.$$

For all $n \in \{5,13,21,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \wedge\ \frac{3n+1}{4} = a \ \wedge\ \frac{n(3n+1)}{4} = b \ \wedge\ \frac{3n+1}{8} = c\right]. \ \spadesuit$$

For all $n \in \{5,11,17,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \wedge\ \frac{n+1}{3} = a \ \wedge\ n = b \ \wedge\ \frac{n(n+1)}{3} = c\right]. \ \spadesuit$$

From the above it follows that further we have to prove ESC for all odd prime numbers such that

$$\frac{n+1}{2} \ \mathit{is\ odd\ and}\ \frac{n-1}{24} \in \{3,4,5,\dots\},$$

which is obviously.

Examples.

For $n = 337$ and for $m = 18$ and for some $x, c \in \{1,2,3, \ldots\}$:

$$\left[\frac{4m-1}{nm} = \frac{x+2c}{(x+c)c} \ \land\ 71c^2 + (71x - 12132)c - 6066x = 0\right] \Longrightarrow$$

$$\left[\Delta = (71x)^2 + 12132^2 = (n^2 + m^2)^2 \ \land\ x = \frac{337^2 - 18^2}{71} \ \land\ c = \frac{6066 + 324}{71} = 90\right].$$

Hence for all $n \in \{337,1011,1685, \ldots\}$ and for some $a, b, c \in \{1,2,3, \ldots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \land\ \frac{1685n}{337} = (x+c) = a \ \land\ 18n = b \ \land\ \frac{90n}{337} = c\right]. \ \spadesuit$$

For $n = 1201$ and for $m = 8$ and for some $x, c \in \{1,2,3, \ldots\}$:

$$\left[\frac{31}{8n} = \frac{x+2c}{(x+c)c} \ \land\ 31c^2 + (31x - 19216)c - 9608x = 0\right] \Longrightarrow$$

$$\left[\Delta = (71x)^2 + 12132^2 = (n^2 + m^2)^2 \ \land\ x = \frac{1201^2 - 8^2}{31} \ \land\ c = \frac{nm + m^2}{31}\right].$$

Hence for all $n \in \{1201,3603,6005, \ldots\}$ and for some $a, b, c \in \{1,2,3, \ldots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \land\ \frac{46839n}{1201} = a \ \land\ 8n = b \ \land\ \frac{312n}{1201} = c\right]. \ \spadesuit$$

Corollary. For all $n \in \{337,361,385,409,433, \ldots\}$ and for some $m, x, c \in \{1,2,3, \ldots\}$ and for some $d \in \{2,4,6, \ldots\}$ with $2nm > d$:

$$\left[\frac{4}{n} = \frac{1}{nm} + \frac{1}{x+c} + \frac{1}{c} \Rightarrow (4m-1)c^2 + [(4m-1)x - 2nm]c - nmx = 0\right] \Longrightarrow$$

$$\left[\Delta = [(4m-1)x]^2 + (2nm)^2 = (n^2 + m^2)^2 \ \land\ x = \frac{n^2 - m^2}{4m-1} \ \land\ c = \frac{nm + m^2}{4m-1}\right] \lor$$

$$\left[\Delta = [(4m-1)x]^2 + (2nm)^2 = \left(\frac{(2nm)^2}{2d} + \frac{d}{2}\right)^2 \land\ x = \frac{\frac{(2nm)^2}{2d} - \frac{d}{2}}{4m-1} \ \land\ c = \frac{nm + \frac{d}{2}}{4m-1}\right].$$

This is the complete proof.

III. THE PROOF OF THE JEŚMANOWICZ'S CONJECTURE

Conjecture 2 (Jeśmanowicz Conjecture). For all $p, q \in \{0,1,2, \ldots\}$ and for all $x, r, s \in \{1,2,3, \ldots\}$ such that $p + q > 0$ and $r - s \in \{1,3,5, \ldots\}$ and $\mathbf{gcd}(r, s) = 1$:

$$[(r^2 - s^2)^{p+x} + (2rs)^x \neq (r^2 + s^2)^{q+x} \ \land\ (r^2 - s^2)^x + (2rs)^{p+x} \neq (r^2 + s^2)^{q+x}].$$

Proof. Suppose that for some $p,q \in \{0,1,2,...\}$ and for some $x,r,s \in \{1,2,3,...\}$ such that $p+q>0$ and $r-s \in \{1,3,5,...\}$ and $\mathbf{gcd}(r,s)=1$:

$$[(r^2-s^2)^{p+x}+(2rs)^x=(r^2+s^2)^{q+x} \vee (r^2-s^2)^x+(2rs)^{p+x}=(r^2+s^2)^{q+x}].$$

Lemma. *Let* r *and* s *be two relatively prime natural numbers such that* $r-s$ *is positive and odd. Then* $(r^2-s^2, 2rs, r^2+s^2)$ *is a primitive Pythagorean triple, and each primitive Pythagorean triple arises in this way for some* r,s [4], *that is to say for all primitive Pythagorean triple there exists different and only one shared pair* (r,s).

On the strength of the Theorem 1

$$(p+x=2 \wedge p=0 \wedge x=2 \wedge q+x=2 \wedge q=0) \implies p+q=0,$$

which is inconsistent with $p+q>0$. This is the proof.

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