THE REFUTATION OF THE ERDŐS-STRAUS CONJECTURE

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Abstract

The refutation of The Erdős-Straus Conjecture.

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Algebra of Sets, Diophantine Equations, Prime Numbers, Pythagoras Equation, Trinomial Square.

Dedicatory

I Dedicate this to My Wife

I. INTRODUCTION

One important topic in number theory is the study of Diophantine equations, equations in which only integer solutions are permitted. The type of Diophantine equation discussed in this paper concerns Egyptian fractions, which deal with the representation of rational numbers as the sum of three unit fractions. [1]

The Erdős–Straus Conjecture (ESC) concerns the Diophantine Equations.

II. THE PROOF OF THE ERDŐS-STRAUS CONJECTURE

Conjecture (ESC). For all $n \in \{2,3,4,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Proof. For all $n \in \{2,4,6,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ \wedge\ \frac{n}{2} = a \ \wedge\ \frac{n+2}{2} = b \ \wedge\ \frac{n(n+2)}{4} = \mathrm{c}\right]. \ \spadesuit$$

For all $n \in \{3,7,11,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n} = \frac{1}{a} + \frac{1}{c} + \frac{1}{b} \wedge \frac{n+1}{2} = a = c \ \wedge\ \frac{n(n+1)}{4} = b\right]. \ \spadesuit$$

Further for all $n \in \{5,9,13,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$: … .

For all $n \in \{5,13,21,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{3n+1}{4}=a \wedge \frac{n(3n+1)}{4}=b \wedge \frac{3n+1}{8}=c\right]. \spadesuit$$

For all $n \in \{5,11,17,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{n+1}{3}=a \wedge n=b \wedge \frac{n(n+1)}{3}=c\right]. \spadesuit$$

For all $n \in \{3,9,15,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{2n}{3}=a=c \wedge n=b\right]. \spadesuit$$

For all $n \in \{3,17,31,\dots\}$ and for some $c,x \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{2n}+\frac{1}{c}+\frac{1}{c+x} \wedge \frac{4n^2-1}{7}=x \wedge \frac{2n+1}{7}=c\right]. \spadesuit$$

Let $\{n\colon\ n=2k+3 \wedge k \in [11,23,35,\dots]\}=\{25,49,73,97,121,145,169,193,217,241,265,\dots\}$.

On the strength of [1] for all $n \in \{13,33,53,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{3n+1}{10}=a \wedge \frac{n(3n+1)}{4}=b \wedge \frac{3n+1}{2}=c\right].$$

On the strength of [1] for all $n \in \{97,111,125,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{2(n+1)}{7}=a \wedge 2n=b \wedge \frac{2n(n+1)}{7}=c\right].$$

For $n=5$ and for some $m,x,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4m-1}{nm}=\frac{x+2c}{(x+c)c} \wedge nm=4n=b \wedge x+c=a \wedge \frac{3}{4}=\frac{x+2c}{(x+c)c}\right] \Longrightarrow$$

$$\left[3c^2+(3x-8)c-4x=0 \wedge \Delta=(3x)^2+8^2=\left(\frac{64+16}{8}\right)^2 = 10^2 \wedge x=2\right] \Longrightarrow$$

$$c=\frac{2+10}{6}=2.$$

Hence for all $n \in \{5,15,25,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{4n}{5}=a \wedge 4n=b \wedge \frac{2n}{5}=c\right]. \spadesuit$$

For all $n \in \{7,21,35,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{4n}{7}=a=c \wedge 2n=b\right]. \spadesuit$$

For $n=7$ and for some $m,x,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4m-1}{nm}=\frac{x+2c}{(x+c)c} \wedge nm=2n=b \wedge x+c=a \wedge \frac{1}{2}=\frac{x+2c}{(x+c)c}\right] \Longrightarrow$$

$$\left[\mathrm{c}^2+(x-4)c-2x=0 \wedge \Delta=x^2+4^2=\left(\frac{16+4}{4}\right)^2 = 5^2 \wedge x=3\right] \Longrightarrow$$

$$c=\frac{1+5}{2}=3.$$

Hence for all $n \in \{7,21,35,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{6n}{7}=a \wedge 2n=b \wedge \frac{3n}{7}=c\right]. \spadesuit$$

For all $n \in \{11,33,55,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{6n}{11}=a=c \wedge 3n=b\right]. \spadesuit$$

For all $n \in \{13,39,65,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{10n}{13} =a \wedge 10n=b \wedge \frac{5n}{13} =c\right]. \spadesuit$$

For all $n \in \{17,51,85,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge 6n=a \wedge n=b \wedge \frac{6n}{17} =c\right]. \spadesuit$$

For all $n \in \{19,57,95,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{10n}{19} =a=c \wedge 5n=b\right]. \spadesuit$$

For all $n \in \{23,69,115,\dots\}$ and for some $a,c,b \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{12n}{23} =a=c \wedge 6n=b\right]. \spadesuit$$

For all $n \in \{29,87,145,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{3}{n}=\frac{1}{a}+\frac{1}{c} \wedge 10n=a \wedge n=b \wedge \frac{10n}{29} =c\right]. \spadesuit$$

For all $n \in \{31,93,155,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge \frac{31}{8n}=\frac{1}{a}+\frac{1}{c} \wedge \frac{16n}{31}=a=c \wedge 8n=b\right]. \spadesuit$$

For $n=241$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left(2n=b=482=\frac{7ac}{a+c} \wedge a=x+c\right) \Longrightarrow 7c^2+(7x-4n)c-2xn=0 \Longrightarrow$$

$$\Delta=(7x)^2+964^2=\left(\frac{964^2+4}{4}\right)^2 \Longrightarrow x=33189.$$

Thus

$$\left(c=\frac{-231359+232325}{14}=\frac{69n}{241} \wedge a=33189+c=33258=138n\right).$$

Hence for all $n \in \{241,723,1205,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\left[\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \wedge 138n=a \wedge 2n=b \wedge \frac{69n}{241}=c\right]. \spadesuit$$

Every odd number bigger than 2 has odd prime divisor, hence sufficient that we prove ESC for odd prime numbers n.

II. The Refutation Of The Erdős-Straus Conjecture

For $n=1201$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\frac{4}{n} \neq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}. \blacksquare$$

This is the proof.

References

[1] http://www.cs.cmu.edu/~avelingk/papers/erdos_straus.pdf