The Amazing Refutation of The Erdős-Straus Conjecture

Leszek W. Guła

Lublin-POLAND

lwgula@wp.pl

28-31 May 2014 – 28-31 December 2014

Abstract

In this paper we present the refutation of The Erdős-Straus Conjecture.

MSC: Primary 11D45, Secondary 11A41.

Keywords

Algebra of Sets, Diophantine Equations, Elementary Number Theory, Erdős-Straus Conjecture, Prime Numbers.

Dedicatory

I Dedicate this to My Wife

I. Introduction

The search for expansions of rational numbers as sums of unit fractions dates to the mathematics of ancient Egypt, in which Egyptian fraction expansions of this type were used as a notation for recording fractional quantities. The Egyptians produced tables such as the Rhind Mathematical Papyrus 2/n table of expansions of fractions of the form 2/n, most of which use either two or three terms. Egyptian fractions typically have an additional constraint, that all of the unit fractions be distinct from each other, but for the purposes of the Erdős–Straus conjecture this makes no difference: if 4/n can be expressed as a sum of at most three unit fractions, it can also be expressed as a sum of at most three distinct unit fractions. [1]

The Erdős–Straus Conjecture concerns the Diophantine Equations.

One important topic in number theory is the study of Diophantine equations, equations in which only integer solutions are permitted. The type of Diophantine equation discussed in this paper concerns Egyptian fractions, which deal with the representation of rational numbers as the sum of three unit fractions. [2]

II. The Proof Of The Erdős-Straus Conjecture?

Conjecture (Erdős– Straus Conjecture). For all $n \in \{2,3,4,\dots\}$ and for some $a,b,c \in \{1,2,3,\dots\}$:

$$\frac{4}{n}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Proof. For all $n \in \{2,3,4,\ldots\}$ and for some $a,b,c \in \{1,2,3,\ldots\}$:

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Longrightarrow 4abc = n(ab + ac + bc). \qquad (1)$$

For $n = 2$ and for $a = 1$ and for $b = c = 2$:

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

For all $n \in \{4,6,8,\ldots\}$ and for some $a,b,c \in \{1,2,3,\ldots\}$:

$$\left[\frac{4}{n} = \frac{1}{\frac{n}{2}} + \frac{1}{\frac{n+2}{2}} + \frac{1}{\frac{n(n+2)}{4}} \ \wedge\ \frac{n}{2} = a \ \wedge\ \frac{n+2}{2} = b \ \wedge\ \frac{n(n+2)}{4} = \mathrm{c}\right]. \ \spadesuit$$

From (1) it follows that for all $n \in \{3,5,7,\ldots\}$ and for some $m,a,c \in \{1,2,3,\ldots\}$:

$$n = \frac{(4m-1)ac}{m(a+c)} \ \wedge\ nm = b \ \wedge\ \mathbf{gcd}(n, 4m-1) \geq 1. \qquad (2)$$

On the strength of (2) for all $n \in \{3,9,15,21,27,33,39,45,51,57,63,69,75,81,87,\ldots\}$ and for some $b,m,a,c \in \{1,2,3,\ldots\}$:

$$\left[n = b \ \wedge\ m = 1 \ \wedge\ n = \frac{(4m-1)ac}{m(a+c)} = \frac{3a}{2} \ \wedge\ \frac{2n}{3} = a = c\right] \Longrightarrow$$

$$\frac{4}{n} = \frac{1}{\frac{2n}{3}} + \frac{1}{n} + \frac{1}{\frac{2n}{3}}.$$

On the strength of (2) for all $n \in \{5,15,25,35,45,55,65,75,85,95,105,115,125,\ldots\}$ and for some $b,m,a,c \in \{1,2,3,\ldots\}$:

$$\left[4n = b \ \wedge\ m = 4 \ \wedge\ n = \frac{(4m-1)ac}{m(a+c)} = \frac{5\mathrm{c}}{2} \ \wedge\ a = 2c = 2\frac{2n}{5}\right] \Longrightarrow$$

$$\frac{4}{n} = \frac{1}{\frac{4n}{5}} + \frac{1}{4n} + \frac{1}{\frac{2n}{5}}.$$

On the strength of (2) for all $n \in \{5,11,17,23,29,35,41,47,53,59,65,71,77,83,89,\ldots\}$ and for some $b,m,a,c \in \{1,2,3,\ldots\}$:

$$\left[n = b \ \wedge\ m = 1 \ \wedge\ n = \frac{(4m-1)ac}{m(a+c)} = \frac{3na}{n+1} \ \wedge\ na = n\frac{n+1}{3} = c\right] \Longrightarrow$$

$$\frac{4}{n} = \frac{1}{\frac{n+1}{3}} + \frac{1}{n} + \frac{1}{\frac{n(n+1)}{3}}.$$

On the strength of (2) for all $n \in \{3,7,11,15,19,23,27,31,35,39,43,47,51,55,59,\ldots\}$ and for some $m,b,a,c \in \{1,2,3,\ldots\}$:

$$\left[n = 4m - 1 \ \wedge\ nm = \frac{n(n+1)}{4} = b \ \wedge\ 2m = \frac{n+1}{2} = a = c\right] \Longrightarrow$$

$$\frac{4}{n} = \frac{1}{\frac{n+1}{2}} + \frac{1}{\frac{n(n+1)}{4}} + \frac{1}{\frac{n+1}{2}}.$$

On the strength of (2) for all $n \in \{7,21,35,49,63,77,91,105,119,133,147,161,\ldots\}$ and for some $m,b,a,c \in \{1,2,3,\ldots\}$:

$$\left[nm = 2n = b \ \wedge\ a = c \ \wedge\ n = \frac{(4m-1)ac}{m(a+c)} = \frac{7a}{4} \ \wedge\ \frac{4n}{7} = a = c\right] \Longrightarrow$$

$$\frac{4}{n} = \frac{1}{\frac{4n}{7}} + \frac{1}{2n} + \frac{1}{\frac{4n}{7}}.$$

On the strength of (2) for all $n \in \{5,13,21,29,37,45,53,61,69,77,85,93,101,109,\ldots\}$ and for some $m,b,a,c \in \{1,2,3,\ldots\}$:

$$\left[nm = b \ \wedge\ m = a = 2c \ \wedge\ n = \frac{(4m-1)ac}{m(a+c)} = \frac{8c-1}{3} \ \wedge\ \frac{3n+1}{8} = c\right] \Longrightarrow$$

$$\frac{4}{n} = \frac{1}{\frac{3n+1}{4}} + \frac{1}{\frac{n(3n+1)}{4}} + \frac{1}{\frac{3n+1}{8}}.$$

On the strength of (2) for all $n \in \{11,33,55,77,99,121,143,165,187,209,231,\ldots\}$ and for some $b,m,a,c \in \{1,2,3,\ldots\}$:

$$\left[nm = b \ \wedge\ m = 3 \ \wedge\ n = \frac{(4m-1)ac}{m(a+c)} = \frac{11a}{6} \ \wedge\ a = c = \frac{6n}{11}\right] \Longrightarrow$$

$$\frac{4}{n} = \frac{1}{\frac{6n}{11}} + \frac{1}{3n} + \frac{1}{\frac{6n}{11}}.$$

III. THE REFUTATION OF THE ERDŐS-STRAUS CONJECTURE

Example. For $n = 9$:

$$\left[9 = \frac{(4m-1)ac}{m(a+c)} = \frac{(4\cdot 1-1)ac}{a+c} \ \vee\ 9 = \frac{(4m-1)ac}{m(a+c)} = \frac{(4\cdot 7-1)ac}{7(a+c)}\right] \Longrightarrow$$

$$\left[\left(3 = \frac{ac}{a+c} \ \wedge\ b = 9 \ \wedge\ a = c = 6\right) \vee \left(\frac{ac}{a+c} = \frac{7}{3} \ \wedge\ b = 63 \ \wedge\ a = c - x\right)\right] \Longrightarrow$$

$$\left[\ \frac{4}{9}=\frac{1}{6}+\frac{1}{9}+\frac{1}{6}\equiv 1\ \vee\ 3(c-x)c=7(c-x)+7c\ \right]\Longrightarrow$$

$$3c^2-(3x+14)c+7x=0\Longrightarrow$$

$$\Delta=(3x+14)^2-84x=(3x)^2+14^2=50^2\Longrightarrow x=16.$$

Hence

$$\left(\ c=\frac{62-50}{6}=2\ \wedge\ a=-14\ \ \wedge\ \frac{4}{9}=\frac{1}{-14}+\frac{1}{63}+\frac{1}{2}\ \right),$$

which is inconsistent with $a\in\{1,2,3,\dots\}$.

But in the first case

$$\frac{4}{9}=\frac{1}{6}+\frac{1}{9}+\frac{1}{6}\equiv 1.$$

This is the example.

Notice that for two prime numbers $73\ \ and\ \ 97$:

$$73=\frac{3\cdot 97+1}{4}.$$

The case 1. On the strength of the above example it must be:

$$n=97=\frac{(4m-1)ac}{m(a+c)}=\frac{(4\cdot 73-1)ac}{73(a+c)}=\frac{291ac}{73(a+c)}\ \Longrightarrow\ \frac{ac}{a+c}=\frac{73}{3}.$$

We have:

$$\left(b=97\cdot 73\ \wedge\ \frac{ac}{a+c}=\frac{73}{3}\ \wedge\ \mathrm{a}\neq c\ \wedge\ a=c-x\right)\Longrightarrow$$

$$3(c-x)c=73(c-x)+73c\ \Longrightarrow 3c^2-(3x+146)c+73x=0\Longrightarrow$$

$$\Delta=(3x)^2+146^2=\left[\ 2\left(37^2+36^2\right)\right]^2\Longrightarrow x=1776.$$

Hence

$$c=\frac{5474-2(37^2+36^2)}{6}=24\ \wedge\ a=c-x=24-1776<0,$$

which is inconsistent with $a\in\{1,2,3,\dots\}$.

The case 2. On the strength of the above example it must be:

$$n = 97 = \frac{(4m-1)ac}{m(a+c)} = \frac{(4\cdot 1-1)ac}{a+c} = \frac{3ac}{a+c} \Longrightarrow \frac{ac}{a+c} = \frac{97}{3}.$$

We have:

$$\left(b = 97 \land \frac{ac}{a+c} = \frac{97}{3} \land \mathrm{a} \neq c \land a = c - x\right) \Longrightarrow$$

$$3(c-x)c = 97(c-x) + 97c \Longrightarrow 3\mathrm{c}^2 - (3x+194)c + 97x = 0 \Longrightarrow$$

$$\Delta = (3x)^2 + 194^2 = \left[2\left(49^2 + 48^2\right)\right]^2 \Longrightarrow x = 3136.$$

Hence

$$c = \frac{9602 - 2(49^2 + 48^2)}{6} = 32 \land a = c - x = 32 - 3136 < 0,$$

which is inconsistent with $a \in \{1,2,3,\dots\}$.

This is the amazing refutation of The Erdős-Straus Conjecture.

REFERENCES

[1] http://en.wikipedia.org/wiki/Erd%C5%91s%E2%80%93Straus_conjecture

[2] http://www.cs.cmu.edu/~avelingk/papers/erdos_straus.pdf